

RELATIVE ORBIFOLD GROMOV-WITTEN THEORY AND DEGENERATION FORMULA

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CONTENTS

1. Introduction	1
2. Review of relative Gromov-Witten theory	5
2.1. Relative pairs and degeneration	5
2.2. Moduli space of relative stable maps	8
3. Review of orbifold Gromov-Witten theory	11
3.1. Basic orbifold theory	11
3.2. Moduli space of stable orbifold morphisms	18
4. Moduli space of relative orbifold stable maps	20
4.1. Orbifold relative pairs and degenerations	20
4.2. Fractional contact order	21
4.3. Stable relative orbifold morphisms to (G, Z)	22
4.4. Stable morphisms to $G^+ \wedge_Z G^-$	25
4.5. Compactness	25
5. Virtual fundamental cycles	30
5.1. The Kuranishi structure	30
5.2. Kuranishi structure at top stratum	31
5.3. Local Kuranishi structure for lower strata	35
5.4. Virtual fundamental cycles and relative invariants	39
6. The degeneration formula	40
Appendix A. Gluing	42
A.1. Set-up	43
A.2. Local homeomorphism of gluing maps	47
References	47

1. INTRODUCTION

The computation of the Gromov-Witten theory is known to be a difficult problem in geometry and physics. There are two major techniques, localization and the degeneration formula. The later was first

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invented by Li-Ruan [LR] (see [IP] for a different version and [Li] for an algebraic treatment). It applies to the situation that a symplectic or Kahler manifold X degenerates to a union of two pieces X^\pm glued along a common divisor Z , which is denoted by $X^+ \wedge_Z X^-$ in this paper. Then, the degeneration formula asserts that the Gromov-Witten invariants of X can be expressed in terms of relative Gromov-Witten invariants of the relative pairs (X^\pm, Z) . During last ten years, the orbifold Gromov-Witten theory has occupied a central place in the Gromov-Witten theory. It is natural to generalize the degeneration formula to the orbifold setting. We will accomplish it in this paper. In a sequel [CLZ], we will apply the degeneration formula established in this paper to prove the invariance of orbifold quantum cohomology under orbifold flops in the complex dimension three. The later settles a famous conjecture of Ruan-Wang for this class of examples.

Let us first recall the main elements of orbifold Gromov-Witten theory (see details §3). Let \mathbf{G} be a symplectic orbifold groupoid with a tamed almost complex structure J . One can associate it a so called *inertia* orbifold $\wedge \mathbf{G}$, which is decomposed into so called *sectors* $\wedge \mathbf{G} = \bigsqcup_{(g)} \mathbf{G}_{(g)}$, where $\mathbf{G}_{(g)}$ is the sector with the monodromy (g) . The Chen-Ruan cohomologies are defined as $H_{CR}(\mathbf{G}, \mathbb{C}) = H^*(\wedge \mathbf{G}, \mathbb{C})$ with appropriate degree shifting. One can define the moduli space $\overline{\mathcal{M}}_{g,m,A}(\mathbf{G})$, $A \in H_2(|\mathbf{G}|, \mathbb{Z})$ of stable orbifold morphisms to \mathbf{G} that represent A . There are evaluation maps $ev_i : \overline{\mathcal{M}}_{g,m,A}(\mathbf{G}) \rightarrow \wedge \mathbf{G}$. The orbifold Gromov-Witten invariants are defined as

$$\langle \tau_{l_1}(\alpha_1), \dots, \tau_{l_m}(\alpha_m) \rangle_{g,A}^{\mathbf{G}} = \int_{[\overline{\mathcal{M}}_{g,m,A}(\mathbf{G})]^{vir}} \prod_i ev_i^* \alpha_i \psi_i^{l_i},$$

where $\alpha_i \in H_{CR}^*(\mathbf{G}, \mathbb{C})$ and ψ_i is the the first Chern class of cotangent line bundle at i -th marked point. The moduli space $\overline{\mathcal{M}}_{g,m,A}(\mathbf{G})$ can be decomposed into disjoint components by specifying the monodromies (or the corresponding twisted sectors) at the marked points. Suppose (g_i) is the monodromy at i -th marked point and set $\mathbf{g} = (g_1, \dots, g_m)$. Then we have the component, denoted by $\overline{\mathcal{M}}_{g,\mathbf{g},A}(\mathbf{G})$. Similarly, we have the invariants $\langle \tau_{l_1}(\alpha_1), \dots, \tau_{l_m}(\alpha_m) \rangle_{g,A}^{\mathbf{G}}$, where $\alpha_i \in H^*(\mathbf{G}_{(g_i)})$.

In the relative setting, we have an additional symplectic divisor $\mathbf{Z} \subset \mathbf{G}$ and we choose an almost complex structure J tamed to the pair (\mathbf{G}, \mathbf{Z}) . We have two types of marked points, *absolute* and *relative marked points*. The absolute marked points are the ones from absolute theory, we may assign each of them a monodromy (g) of \mathbf{G} . For each relative marked point, we attach it a monodromy (h) of \mathbf{Z} and a *fractional contact order* $\ell = k/|h|$. We explain its meaning. Suppose that

we have a local holomorphic orbifold morphism

$$f : \mathbb{D}/\mathbb{Z}_m \rightarrow (V \times \mathbb{C})/G_z,$$

where \mathbb{D}/\mathbb{Z}_m is an orbifold disc, $z = f(0)$, $V/G_z \subset \mathbf{Z}$ is a neighborhood of z and $(V \times \mathbb{C})/G_z$ represents the neighborhood of z in \mathbf{G} . f being an orbifold morphism means that we can lift f to an equivariant map

$$f_0 = (f^1, f^2) : \mathbb{D} \rightarrow V \times \mathbb{C}$$

which is equivariant with respect to an injective morphism $\psi : \mathbb{Z}_m \rightarrow G_z$ that sends the generator $e^{2\pi i/m}$ to h . Then, k in the formula is the lowest degree of f^2 in its Taylor expansion, the contact order in smooth case. ℓ can also be understood via the Thom form (cf. §4.2). Suppose that $f : \mathbb{C} \rightarrow \mathbf{G}$ is a global holomorphic orbifold morphism such that the image of f intersects \mathbf{Z} only at (finite) relative marked points. One can show that the (orbifold) homological intersection $f^*[Z] \cap [C] = [Z] \cap f_*[C]$ is the sum of fractional contact orders (Lemma 4.3).

Suppose that we have m absolute marked points and k relative marked points, let (\mathbf{g}) and (\mathbf{h}) be the collection of absolute and relative monodromies. Let $T_k = (\ell_1, \dots, \ell_k)$ be the collection of contact orders and it is a partition of $[Z] \cap f_*[C]$. Similarly, we can define the moduli space of stable relative orbifold morphisms $\overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z})$ (see §4 for the details). Our main theorem is

Theorem 1.1. *$\overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z})$ is compact and carries a virtual fundamental cycle.*

There are two types of evaluation maps. For each absolute marked point,

$$ev_i : \overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z}) \rightarrow \mathbf{G}_{(g_i)}, \quad 1 \leq i \leq m,$$

and for each relative marked point,

$$ev_j^r : \overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z}) \rightarrow \mathbf{Z}_{(h_j)}, \quad 1 \leq j \leq k.$$

Let $\alpha_i \in H^*(\mathbf{G}_{(g_i)})$, $\beta_j \in H^*(\mathbf{Z}_{(h_j)})$ and $\mathcal{T}_k = ((\ell_1, \beta_1), \dots, (\ell_k, \beta_k))$. The relative orbifold Gromov-Witten invariants is defined as

$$\left\langle \prod_{i=1}^m \tau_{l_i}(\alpha_i) | \mathcal{T}_k \right\rangle_{g,A}^{\mathbf{G}, \mathbf{Z}} = \int_{[\overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z})]^{vir}} \prod_i ev_i^* \alpha_i \psi_i^{l_i} \prod_j ev_j^{r,*} \beta_j.$$

In fact, these invariants are independent of a particular orbifold groupoid representation and are invariants of the underline orbifold.

Suppose that \mathbf{G} is degenerated to $\mathbf{G}^+ \wedge_{\mathbf{Z}} \mathbf{G}^-$. Our main formula is the following *degeneration formula*

$$(1.1) \quad \sum_{A \in [A]} \langle \mathbf{a} \rangle_{g,A}^{\bullet \mathbf{G}_t} = \sum_{\Gamma} \sum_I C(\Gamma, I) \langle \mathbf{a}^+ | \mathbf{b}^I \rangle_{\Gamma^+}^{\bullet(\mathbf{G}^+, \mathbf{Z})} \langle \mathbf{a}^- | \mathbf{b}_I^* \rangle_{\Gamma^-}^{\bullet(\mathbf{G}^-, \mathbf{Z})}.$$

We refer the reader to §6 for the meaning of symbols.

The technique of this paper is similar to that of the smooth case [LR]. In fact, the analysis is identical, which we will review in the appendix. The new ingredients are the global properties of orbifold structures. This properties was called *compatible system* by Chen-Ruan [CR1] when they introduced the orbifold Gromov-Witten theory. During the recent year, the preferred treatment is to package it into the language of groupoid and stack. We follow this approach.

The paper is organized as follows. We review the relative Gromov-Witten theory in §2 and orbifold Gromov-Witten theory in §3 to set up notations. In particular, in §3 we take the opportunity to review the set-up of orbifold theory by the language of groupoid. The core of the paper is §4 and §5. In §4, we introduce the moduli space of stable relative orbifold morphism. One of the highlight is the compactness theorem. In §5, we construct the virtual fundamental cycle. There are several approaches [FO], [LT], [R2], [H]. We use the Kuranishi structure in the Fukaya-Ono's approach ([FO]). The degeneration formula then follows quickly in §6.

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We make several remarks on the recent paper [AF] by Abramovich-Fantechi by making some comparisons with this paper.

- In [AF], in order to develop orbifold techniques in studying the degeneration of Gromov-Witten theory, the authors define the relative orbifold Gromov-Witten invariants in the algebraic geometry sense. Their degeneration formula (§0.4[AF]) is same as ours (§6): for instance, the d_j in their formula is the intersection multiplicity ℓ in our paper;
- when considering an orbifold pair (X, Z) , apriori, it is not clear how to define Z , for example, in the groupoid sense. This is formulated in §4.1. In fact, the neighborhood of Z can be thought as a Seifert bundle (§4.7 [BG]); in [AF], for the sake of emphasizing the orbifold technique, usually the structure at Z is simplified;
- in both [AGV] and [AF], the authors introduce ghost automorphisms(cf. §1.1.1 [AF]). Such an orbifold structure is captured in Lemma 4.2. We would like to thank Abramovich for pointing out this to us.

2. REVIEW OF RELATIVE GROMOV-WITTEN THEORY

As we mentioned in the introduction, the paper is devoted to develop the *relative orbifold Gromov-Witten theory and its degeneration formula*. This is a generalization of corresponding theory in the smooth case([LR]). In this section, we will review the basic constructions of the relative Gromov-Witten theory.

2.1. Relative pairs and degeneration. We start from the basic geometric construction of the degeneration of algebraic or symplectic manifold. The construction in the smooth setting is now well-known to the experts.

2.1.1. Neighborhood of divisor. A symplectic relative pair (X, Z) is a symplectic manifold (X, ω) together with a symplectic divisor or codimension two symplectic submanifold Z in X . We can standardize the local structure around Z . Pick a compatible almost complex structure on the normal bundle $N := N_{Z|X}$. Then N is a Hermitian line bundle. Its principal S^1 -bundle Y is the unit circle bundle of N where S^1 acts as complex multiplication. Then $N = Y \times_{S^1} \mathbb{C}$.

On Y , there is a connection 1-form θ which is dual to the vector field T generated by the action. Let ω_Z be the symplectic form on Z .

$$(2.1) \quad \omega_o := \pi^* \omega_Z + \frac{1}{2} d(\rho^2 \wedge \theta).$$

defines a form on $N \setminus \{Z\}$. Here, we take Z to be the 0-section, and ρ to be the radius function on \mathbb{C} . This form can be extended over N and it is a symplectic form over N . The S^1 action is Hamiltonian in the sense: $i_T \omega_o = -\frac{1}{2} d\rho^2$.

Let $\mathbb{D}_\epsilon \subset \mathbb{C}$ be the disk of radius ϵ , \mathbb{D} be the unit disk and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. We have the following subbundles of N :

$$\mathbb{D}_\epsilon N = Y \times_{S^1} \mathbb{D}_\epsilon, \quad N^* = Y \times_{S^1} \mathbb{C}^*, \quad \mathbb{D}_\epsilon^* N = Y \times_{S^1} \mathbb{D}_\epsilon^*.$$

The projective completion of N is $Q = Y \times_{S^1} \mathbb{CP}^1$. In algebraic situation, $Q = \mathbb{P}(N \oplus \mathbb{C})$. It contains two special sections: the 0-section and the ∞ -section, denoted by Z_0 and Z_∞ respectively. Both of them are identified with Z .

By the symplectic neighborhood theorem, there exists a neighborhood U of Z such that $(U, \omega) \cong (\mathbb{D}_\epsilon N, \omega_o)$ for some $\epsilon > 0$. Here, ω_o is given in (2.1). We normalize the local structure near Z such that a neighborhood U of Z satisfies

$$(2.2) \quad (U, \omega) \cong (\mathbb{D} N, \omega_o).$$

2.1.2. *Symplectic manifold with cylindric ends.* An equivalent description of a relative pair is a *manifold with cylindric ends*. Let's review the construction. Let Y be as above. A cylinder is $CY_I := Y \times I$ where I is some interval of \mathbb{R} . Define $\overline{CY}_I = Y \times \bar{I}$. Set

$$CY = CY_{\mathbb{R}}, \quad CY^{\pm} = CY_{(0, \pm\infty)}; \quad CY_T = CY_{(-T, T)}, \quad CY_T^{\pm} = CY_{(0, \pm T)}.$$

On CY , we define a symplectic form

$$(2.3) \quad \omega_c = \pi^* \omega_Z + d(\theta \wedge t).$$

Then the Hamiltonian action is given by the Hamiltonian function $H(y, t) = t$.

It is easy to see $N^* \cong Y \times \mathbb{R}$. (In this paper, by \cong , we mean biholomorphic.) The induced symplectic form on CY from ω_o is

$$\hat{\omega}_o = \pi^* \omega_Z + \frac{1}{2} d(e^{2t} \wedge \theta).$$

$\hat{\omega}_o \neq \omega_c$, however, they are different up to a deformation. Similarly, $\mathbb{D}^*N \cong CY^-$, $N \setminus \overline{\mathbb{D}N} \cong CY^+$.

Recall that Y is a space with S^1 -action. Let \bar{Y} denote the space Y with the reverse S^1 -action. Then

$$(2.4) \quad CY_T^+ \cong C\bar{Y}_T^-$$

by $(y, t) \rightarrow (y, -t)$. Let \bar{N} be the line bundle corresponding to $C\bar{Y}$. Then

$$(2.5) \quad \bar{N} = N^{-1}.$$

Now, we consider $X \setminus Z$. Set $X_{\emptyset} = X \setminus \mathbb{D}N$. Then $X \setminus Z \cong X_{\emptyset} \cup \mathbb{D}^*N$. Replacing \mathbb{D}^*N by CY^- , we obtain a manifold with cylindric ends. Set

$$X^Z = X_{\emptyset} \cup CY^-, \quad X_T^Z = X_{\emptyset} \cup CY_T^-.$$

For simplicity, we denote them X^* and X_T^* respectively.

It is clear that we can reverse the constructions to obtain a relative pair from a manifold with a cylindric end.

2.1.3. *Degeneration.* Suppose that two symplectic manifolds X^+ and X^- intersect at a common divisor Z . We say that the intersection is a *normal crossing* if the normal bundles N^{\pm} of Z in X^{\pm} satisfy $N^+ = (N^-)^{-1}$. We call such an intersection pair a *degenerated* symplectic manifold and denote it by

$$(2.6) \quad X = X^+ \wedge_Z X^-.$$

Similarly, there is a cylindric model for X

$$X^* = CY^- \cup X_{int} \cup CY^+.$$

Here, $X_{int} = X_0 - (\mathbb{D}N^+ \cup \mathbb{D}N^-)$. X_{int} may consists of more than two components.

From X , we are able to construct a family of symplectic manifolds $X_{T,\theta}$ for any parameter of a pair (T, θ) , where $T \geq 0$ and $\theta \in S^1$. In fact, we obtain $X_{T,\theta}$ by gluing two cylinders CY_{2T}^- and CY_{2T}^+ in X^* via the identification $(y, t) \rightarrow (\theta \cdot y, t + 2T)$. The set of parameters of the family is $[0, \infty) \times S^1$. By taking $t = \exp(-T + i\theta)$, the set of parameters is then identified with the punctured disk, denoted by \mathfrak{D}^* . Indeed it is known that

Proposition 2.1. *There is a smooth family of symplectic manifolds (\mathcal{D}, ω) and a projection*

$$(2.7) \quad \pi : (\mathcal{D}, \omega) \rightarrow \mathfrak{D}$$

such that $\pi^{-1}(0) = X$ and $\pi^{-1}(t) \cong X_{T,\theta}$.

A neighborhood of Z in \mathcal{D} is $\hat{X} = \mathbb{D}N^+ \wedge_Z \mathbb{D}N^-$. Set

$$\hat{\mathcal{D}} = \mathbb{D}N^+ \oplus \mathbb{D}N^-, \quad \hat{\omega}_f = \omega_o^+ \oplus \omega^-.$$

The key observation is that $N^+ \otimes N^-$ is a trivial bundle over Z . Hence there is a natural projection

$$\pi' : N^+ \otimes N^- \cong Z \times \mathbb{C} \rightarrow \mathbb{C}.$$

The projection $\pi : \hat{\mathcal{D}} \rightarrow \mathfrak{D}$ is defined to be the composition of maps

$$\pi : N^+ \oplus N^- \xrightarrow{\otimes} N^+ \otimes N^- \xrightarrow{\pi'} \mathbb{C}.$$

Remark 2.2. *We describe the glued manifolds $X_t = \pi^{-1}(t)$ from X . The reverse process from X_t to X is called the symplectic cutting (see [LR]).*

2.1.4. Degeneration of X along Z . A particular important example is so-called *the degeneration to the normal cone* or the degeneration of X along Z . It appears in the definition of the relative stable map. We review the construction.

Recall that $Q = \mathbb{P}(N \oplus \mathbb{C})$ is the projective completion of the normal bundle $N_{Z|X}$ with a zero section Z_0 and an infinity section Z_∞ . For any non-negative integer m , construct Q_m by gluing together m copies of Q , where the infinity section of the i^{th} component is glued to the zero section of the $(i-1)^{th}$ component for $2 \leq i \leq m$. Denote the zero section of the i^{th} component by $Z_{i,0}$, and the infinity section by $Z_{i,\infty}$, so the singular set of Q_m is

$$\text{Sing}(Q_m) = \bigcup_{i=1}^{m-1} Z_{i,0} = \bigcup_{i=2}^m Z_{i,\infty}.$$

Define X_m by gluing X to Q_m along $Z \subset X$ and $Z_{1,\infty} \subset Q_m$. In particular, $X_0 = X$ will be referred to as the zero level and the i -th component of Q as the level i rubble components. Write $Z = Z_{0,0}$. Then $\text{Sing}(X_m) = \cup_{i=0}^{m-1} Z_{i,0}$.

Let $\text{Aut}_m^{\text{rel}} := \text{Aut}(Q_m, \text{Sing}(Q_m))$ be the group of automorphisms of Q_m preserving $\text{Sing}(Q_m)$. Then $\text{Aut}_m^{\text{rel}} \cong (\mathbb{C}^*)^m$, where each factor of $(\mathbb{C}^*)^m$ dilates the fibers of the i -th \mathbb{P}^1 -bundle. The group acts both on Q_m and X_m .

One can remove $\text{Sing}(Q_m)$ (resp. $\text{Sing}(X_m)$) from Q_m (resp. X_m) and change the complement to a series of cylinders (resp. manifolds with cylinder ends). Then, we obtain the cylindric models Q_m^* (resp. X_m^*).

2.2. Moduli space of relative stable maps.

2.2.1. Relative stable maps. We start with the moduli space of stable maps. Suppose that (X, ω) is a compact symplectic manifold and J is a *tamed almost complex structure*. Namely, $\omega(v, Jv) > 0$ for any nonzero tangent vector v .

Definition 2.1. *A stable (J -holomorphic) map is an equivalence class of pairs (C, f) . Here C is a connected nodal marked Riemann surface and $f : C \rightarrow X$ is a continuous map whose restriction to each component of C (called a component of f in short) is holomorphic. Furthermore, it satisfies the stability condition that the automorphism group is finite.*

Here, (C, f) , (C', f') are equivalent, if there is a biholomorphic map $h : C' \rightarrow C$ such that $f' = f \circ h$.

We define the moduli space $\overline{\mathcal{M}}_{g,m,A}(X)$ to be the set of equivalence classes of stable holomorphic maps such that the homology class of the map $[f]$ is $f_*[C] = A \in H_2(X, \mathbb{Z})$. The virtual dimension of the moduli space is computed by the index theory

$$\text{vir} \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,m,A}(X) = c_1(X)(A) + (3-n)(g-1) + m,$$

where n is the complex dimension of X .

Let (X, Z) be a relative pair. J is called *tamed to (X, Z)* if (i) J is tamed with X , (ii) Z is almost complex, (iii) a neighborhood of Z is standardized as (2.2). The relative GW invariants are defined by counting the number of stable holomorphic maps intersecting Z at finitely many points with prescribed contact orders. More precisely, fix a k -tuple $T_k = (\ell_1, \dots, \ell_k)$ of positive integers, consider a marked pre-stable curve

$$(C, x_1, \dots, x_l, y_1, \dots, y_k)$$

and stable J -holomorphic maps $f : C \rightarrow X$ such that the divisor f^*Z is

$$f^*Z = \sum_{i=1}^k \ell_i y_i.$$

The above definition only makes sense if no component of C is mapped into Z . For general situation, we need to consider the degenerated target spaces X_m .

Now consider a nodal curve C mapped into X_m by $f : C \rightarrow X_m$. We divide the marked and nodal points into absolute and relative types and require:

- (1) the absolute marked and nodal points mapped into $X_m - \text{Sing}(X_m)$;
- (2) the relative marked points mapped into $Z_{m,0}$;
- (3) the relative nodes mapped into $\text{Sing}(Q_m)$;
- (4) $f^{-1}(\cup_{i=0}^m Z_{i,0})$ consists of only relative marked and nodal points;
- (5) the balanced condition that $f^{-1}(Z_{i,\infty} = Z_{i-1,0})$ consists of a union of nodes so that for each node $p \in f^{-1}(Z_{i,\infty} = Z_{i-1,0})$, $i = 1, 2, \dots, m$, the two branches at the node are mapped to different irreducible components of X_m and the orders of contacts to $Z_{i,\infty} = Z_{i-1,0}$ are equal.

An isomorphism of two such J -holomorphic maps f and f' to X_m consists of a diagram

$$\begin{array}{ccc} (C, x_1, \dots, x_l, y_1, \dots, y_k) & \xrightarrow{f} & X_m \\ \phi \downarrow & & \downarrow t \\ (C', x'_1, \dots, x'_l, y'_1, \dots, y'_k) & \xrightarrow{f'} & X_m \end{array}$$

where ϕ is an isomorphism of marked curves and $t \in \text{Aut}_m^{\text{rel}}$. With the preceding understood, a relative J -holomorphic map to X_m is said to be stable if it has only finitely many automorphisms.

Recall that we have a natural map $\pi_m : X_m \rightarrow X$, which is the identity on the root component $X_0 = X$ and contracts all the rubble components to $Z = Z_{0,0}$ via the fiber bundle projections. We define $[f]$ to be $(\pi_m f)_*[C] \in H_2(X, \mathbb{Z})$. It is known that

$$(2.8) \quad \sum_{i=1}^k \ell_i = [f] \cdot Z.$$

2.2.2. Dual graph and stratification. It is well-known that the moduli space of stable maps has a stratification indexed by the combinatorial type of its decorated dual graph. This construction generalizes to the relative setting.

Given a relative stable map, we can assign a (connected) relative graph Γ (called type) consisting of the following data:

- (1) a vertex decorated by $A \in H_2(X; \mathbb{Z})$, genus g and level i of the i -th component in X_m ,
- (2) a tail for each absolute marked point,
- (3) a relative tail decorated by its contact order for each relative marked point,
- (4) an absolute edge for each absolute node,
- (5) a relative edge decorated by its contact order for each relative node.

Furthermore, for a pair of vertices connecting an absolute (resp. relative) edge, their level should equal (resp. different by one). Moreover, the labelled information should be compatible with (2.8). Let $V(\Gamma)$, $E(\Gamma)$ and $T(\Gamma)$ be the sets of vertices, edges and tails of Γ respectively. For each $\nu \in V(\Gamma)$ let g_ν be the (geometric) genus of the component of C corresponding to ν .

Definition 2.2. Let Γ be a dual graph. The genus of Γ is defined as

$$g(\Gamma) = \dim H^1(\Gamma) + \sum_{\nu \in V(\Gamma)} g_\nu.$$

Similarly, the fundamental class A is defined as the sum of homology decorations at each vertex.

For each partition $T_k = (\ell_1, \dots, \ell_k)$ of $Z \cdot A$, let $\mathcal{S}(g, m, A, T_k)$ be the set of relative graph with genus g , the fundamental class A , m -absolute tails, k -relative tails decorated by the partition T_k .

We can define a partial order among relative graphes as follows. Let $\Gamma \in \mathcal{S}(g, m, A, T_k)$. We introduce two types of *contraction*: (i), for an edge e between vertices of the same level, one can contract the edge and modify the vertices and its decorations in an obvious way to obtain another relative graph Γ' ; or (ii) one can also contract all the edges between the level $i, i+1$ vertices and lower the level of vertices of level $j \geq i$ by 1 to obtain a relative graph Γ' . We define a partial order by saying that $\Gamma' \leq \Gamma$ if Γ is obtained from Γ' by a sequence of contractions. There is a unique maximal graph, denoted by Γ_{g,m,A,T_k} , in $\mathcal{S}(g, m, A, T_k)$.

Definition 2.3. Define \mathcal{M}_Γ as the moduli space of relative stable maps of type Γ and $\overline{\mathcal{M}}_\Gamma$ be the union of $\mathcal{M}_{\Gamma'}$ for all $\Gamma' \leq \Gamma$. Define

$$\overline{\mathcal{M}}_{g,m,A,T_k}(X, Z) = \overline{\mathcal{M}}_{\Gamma_{g,m,A,T_k}}.$$

It is clear that we have a stratification

$$\overline{\mathcal{M}}_{g,m,A,T_k}(X, Z) = \bigsqcup_{\Gamma \leq \Gamma(g,m,A,T_k)} \mathcal{M}_\Gamma.$$

The virtual dimension of the moduli space is given by the formula

$$\text{vir dim}_{\mathbb{C}} \overline{\mathcal{M}}_{(g,m,A,T_k)}(X, Z) = c_1(A) + (3-n)(g-1) + m + k - \ell(T_k),$$

where $\ell(T_k) = Z \cdot A = \sum \ell_i$.

There are two types of evaluation maps. For each absolute marked point, we have

$$ev_i : \overline{\mathcal{M}}_{g,m,A,T_k} \rightarrow X.$$

For each relative marked point, we have

$$ev_j^r : \overline{\mathcal{M}}_{g,m,A,T_k} \rightarrow Z.$$

In [LR], a virtual cycle was constructed for the above moduli space. Let $\alpha_i \in H^*(X)$, $\beta_j \in H^*(Z)$. The relative Gromov-Witten invariant is defined as

$$\langle \prod_i^m \tau_{l_i}(\alpha_i) | \mathcal{T}_k \rangle_g^{X,Z} = \frac{1}{|Aut(\mathcal{T}_k)|} \int_{[\overline{\mathcal{M}}]^{vir}} \prod_i ev_i^*(\alpha_i) \psi_i^{l_i} \prod_j ev_j^{r,*}(\beta_j),$$

where $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,m,A,T_k}(X, Z)$, $\mathcal{T}_k = \{(\ell_1, \beta_1), \dots, (\ell_k, \beta_k)\}$ and ψ_i is the first Chern class of cotangent line bundle at the marked point x_i .

3. REVIEW OF ORBIFOLD GROMOV-WITTEN THEORY

In this section, we review the basic construction of the orbifold Gromov-Witten theory developed by Chen-Ruan (see Abramovich-Graber-Vistoli ([AGV]) for algebraic treatment). Chen-Ruan's original treatment used the language of orbifold charts. It become rather clumsy while treating the maps or morphisms between orbifolds. Afterwards, a great deal of efforts was put into clarifying the foundation using the language of groupoid/stack (see a beautiful book [ALR] for the treatment). However, a compactness theorem is still lacking in this setting. Such a compactness theorem will be addressed in §4.5.

3.1. Basic orbifold theory. In this section, we review the basic concepts in the orbifold theory. Our reference is [ALR]. In this paper, a groupoid is denoted by $\mathbf{G}, \mathbf{C}, \mathbf{H}$ and etc.

3.1.1. *Orbifold structure.*

Definition 3.1. A topological groupoid \mathbf{G} consists of a space G_0 of objects and a space G_1 of arrows, together with five continuous structure maps, listed below.

- (1) The source map $s : G_1 \rightarrow G_0$.
- (2) The target map $t : G_1 \rightarrow G_0$.
- (3) If g and h are two arrows with $s(h) = t(g)$, one can form their composition hg with $s(hg) = s(g)$ and $t(hg) = t(h)$. We denote this by $m(g, h) = hg$. Moreover, the composition map m is required to be associative.
- (4) The unit (or identity) map $u : G_0 \rightarrow G_1$ which is a two-sided unit for the composition.
- (5) An inverse map $i : G_1 \rightarrow G_1$, written $i(g) = g^{-1}$.

A Lie groupoid is a topological groupoid \mathbf{G} where G_0 and G_1 are smooth manifolds, and such that the structure maps s, t, m, u, i are smooth. Furthermore, s and t are required to be submersions.

Let \mathbf{G} be a Lie groupoid. For a point $x \in G_0$, the set of all arrows from x to itself is a Lie group, denoted by G_x and called the isotropy or local group of x . The set $ts^{-1}(x)$ is called the orbit of x . The orbit space $|\mathbf{G}|$ of \mathbf{G} is the space of orbits. We call \mathbf{G} a groupoid presentation of $|\mathbf{G}|$.

Definition 3.2. Let \mathbf{G} be a Lie groupoid and s, t be its source and target map. \mathbf{G} is called proper if (s, t) is a proper map. \mathbf{G} is called etale if s and t are local diffeomorphisms. We define an orbifold groupoid to be a proper etale Lie groupoid.

Next we discuss morphisms between groupoids, and natural transformations.

Definition 3.3. Let \mathbf{G} and \mathbf{H} be two Lie groupoids. A homomorphism $\phi : \mathbf{H} \rightarrow \mathbf{G}$ consists of two smooth maps $\phi : H_0 \rightarrow G_0$ and $\phi : H_1 \rightarrow G_1$, which together commute with all the structure maps for the groupoids \mathbf{G} and \mathbf{H} .

Definition 3.4. Let $\phi, \psi : \mathbf{H} \rightarrow \mathbf{G}$ be two homomorphisms. A natural transformation α from ϕ to ψ (notation : $\alpha : \phi \rightarrow \psi$) is a smooth map $\alpha : H_0 \rightarrow G_1$ giving for each $x \in H_0$ an arrow $\alpha(x) : \phi(x) \rightarrow \psi(x)$ in G_1 , natural in x in the sense that for any $g : x \rightarrow x'$ in H_1 the identity $\psi(g)\alpha(x) = \alpha(x')\phi(g)$ holds.

Definition 3.5. A homomorphism $\phi : \mathbf{H} \rightarrow \mathbf{G}$ between Lie groupoids is called an equivalence if

(i) The map

$$t\pi_1 : G_1 \times_{G_0} H_0 \rightarrow G_0$$

defined on the fibered product of manifolds

$$\{(g, y) | g \in G_1, y \in H_0, s(g) = \phi(y)\}$$

is a surjective submersion.

(ii) The square

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi_1} & G_1 \\ (s,t) \downarrow & & (s,t) \downarrow \\ H_0 \times H_0 & \xrightarrow{\phi_0 \times \phi_0} & G_0 \times G_0 \end{array}$$

is a fiber product.

It is clear that a homomorphism $\phi : \mathbf{H} \rightarrow \mathbf{G}$ induces a continuous map $|\phi| : |\mathbf{H}| \rightarrow |\mathbf{G}|$. Moreover, if ϕ is an equivalence, $|\phi|$ is a homeomorphism.

A guiding example is given by an open covering $\{U_\alpha\}_{\alpha \in I}$ of a smooth manifold. For such a covering we can define an orbifold groupoid

$$G_0 = \bigsqcup_{\alpha} U_\alpha, \quad G_1 = \bigsqcup_{\alpha, \beta} U_\alpha \cap U_\beta.$$

In the case of *effective* orbifold ($G_x = 1$ for a generic point x), the above construction generalize to orbifold and one can describe an orbifold structure using an open covering (see Chapter one in [ALR]). However, in non-effective case, the language of charts is insufficient. A guiding example of equivalence is the refinement of open coverings. Occasionally, we simply refer an equivalence as a *refinement*.

Definition 3.6. Two Lie groupoids \mathbf{G} and \mathbf{G}' are said to be Morita equivalent if there exists a third groupoid \mathbf{H} and two equivalences

$$\mathbf{G} \xleftarrow{\phi} \mathbf{H} \xrightarrow{\phi'} \mathbf{G}'.$$

Definition 3.7. An orbifold structure on a paracompact Hausdorff space X consists of an orbifold groupoid \mathbf{G} and a homeomorphism $f : |\mathbf{G}| \rightarrow X$. If $\phi : \mathbf{H} \rightarrow \mathbf{G}$ is an equivalence, then $|\phi| : |\mathbf{H}| \rightarrow |\mathbf{G}|$ is a homeomorphism, and we say that $(\mathbf{H}, f \circ |\phi|)$ defines an equivalent orbifold structure on X .

An orbifold is a space X equipped with a Morita equivalence class of orbifold structures $[\mathbf{X}]$.

A primary example in the orbifold Gromov-Witten theory is the orbifold Riemann surfaces. Let C be a Riemann curve with marked points and nodal points. Let

$$M = \{p_1, \dots, p_m\}, \quad N = \{q_1, \dots, q_n\}$$

be the set of marked points and set of nodal points respectively. For each marked point p_i , we denote the component containing p_i by C_{p_i} . For each nodal point q_j , we denote the components containing it by $C_{q_j}^\pm$, (it is possible that $C_{q_j}^\pm$ are the same component).

Example 3.1 (Orbifold Riemann surfaces). *Let C be as above. By an orbifold structure on C we mean for each p_i (resp. q_j on C_i^\pm) there is a local group G_{p_i} (resp. $G_{q_j}^\pm$) and (since we are working over C) a canonical isomorphism $G_{p_i} \cong \mathbb{Z}_{r_i}$ (resp. $G_{q_j}^\pm \cong \mathbb{Z}_{s_j^\pm}$) for some positive integer r_i (resp. s_j^\pm). A neighborhood of p_i (resp. of q_j in $C_{q_j}^\pm$) is uniformized by the branched covering map $z \rightarrow z^{r_i}$ ($z \rightarrow z^{s_j^\pm}$).*

Moreover, for each nodal point q we require the balance condition. That is, $s^+ = s^- =: s$, and a neighborhood of a nodal point (viewed as a neighborhood of the origin of $\{zw = 0\} \subset \mathbb{C}^2$) is uniformized by a branched covering map $(z, w) \rightarrow (z^s, w^s)$, and with group action $e^{2\pi i/s}(z, w) = (e^{2\pi i/s}z, e^{-2\pi i/s}w)$.

An orbifold structure on C is uniquely specified by r_i and s_j . They are called the multiplicity on each marked and nodal point. We will call C smooth if the underlying curve $|C|$ is smooth, and we will call the orbicurve nodal if $|C|$ is nodal.

An orbi-curve C is an example of effective orbifold. We conveniently choose an open covering (and hence an orbifold groupoid \mathbf{C}) consisting of C^* (the complement of marked and nodal points) and orbifold charts for each marked and nodal point. It is easy to check that an automorphism of C corresponds to an automorphism of underlying curve $|C|$ preserving the multiplicities.

3.1.2. Orbifold bundles. Let $\mathbf{G} = (G_0, G_1)$ be an orbifold groupoid. Let $\pi_0 : E_0 \rightarrow G_0$ be a vector bundle of rank n . Over G_1 we have two bundles $\pi_s : s^*E_0 \rightarrow G_1$ and $\pi_t : t^*E_0 \rightarrow G_1$. Let σ be a section of the bundle

$$\text{Hom}(s^*E_0, t^*E_0) \rightarrow G_1$$

such that $\sigma(g)$ is an isomorphism for any $g \in G_1$ and $\sigma(hg) = \sigma(h)\sigma(g)$. σ induces a set of arrows E_1 on E_0 :

$$E_1 = \{(\alpha, \beta) \in s^*E_0 \times t^*E_0 \mid \beta = \sigma(\pi_s(\alpha))\alpha\}.$$

Then $\mathbf{E} = (E_0, E_1)$ is an orbifold groupoid. We call \mathbf{E} an *orbifold vector bundle over \mathbf{G}* .

Remark 3.2. *In the construction, E_1 is completely determined by σ . In fact, in [ALR], σ is treated as a representation of the action of G_1 on E_0 . Here, we prefer to present the bundle as $\mathbf{E} = (E_0, \sigma)$.*

The same treatment can be applied to general fiber bundles.

Let \mathbf{G} and \mathbf{H} be two orbifold groupoids. Let $f : \mathbf{G} \rightarrow \mathbf{H}$ be a groupoid morphism. Let $\mathbf{E} = (E_0, \sigma)$ be an \mathbf{H} -bundle. It is natural to pull back bundle $f^*\mathbf{E} = (f_0^*E_0, f_1^*\sigma)$, where $f = (f_0, f_1)$.

Let $\mathbf{E} = (E_0, \sigma)$ be a bundle over $\mathbf{G} = (G_0, G_1)$. Let $u : G_0 \rightarrow E_0$ be a section. It is called a *section* of \mathbf{E} if for any $g \in G_1$,

$$\sigma(g)(u(s(g))) = u(t(g)).$$

Let $\Omega(\mathbf{E})$ be the space of sections.

Let u be a section transversal to the 0-section. Let $M_0 = u^{-1}(0) \subset G_0$. Then $M_1 = s^{-1}(M_0)$ gives the set of arrows on M_0 . We obtain a groupoid $\mathbf{M} = (M_0, M_1)$. Hence we conclude that

Lemma 3.3. *Let $\mathbf{E} \rightarrow \mathbf{G}$ be an orbifold vector bundle and s be a transversal section, then $\mathbf{M} = s^{-1}(0)$ has a structure of an orbifold groupoid.*

Let $\mathbf{G} = (G_0, G_1)$ be an orbifold groupoid. We introduce its tangent bundle. Let $E_0 = TG_0$. We now describe σ . For each point $g \in G_1$, by the local diffeomorphism of s and t it induces a linear isomorphism from $T_{s(g)}G_0$ to $T_{t(g)}G_0$. We denote this $\sigma(g)$. Then $T\mathbf{G} := (TG_0, \sigma)$ is the tangent bundle of \mathbf{G} .

Similarly, we can define the cotangent bundle $T^*\mathbf{G}$ and other tensor bundles such as $\Lambda^*T^*\mathbf{G}$, etc. By considering the sections of these bundles, we have all kinds of tensor fields on \mathbf{G} .

Let $\mathbf{E} \rightarrow \mathbf{G}$ be a good vector bundle (Definition 2.28 [ALR]). We can define a metric h on \mathbf{E} . In fact, this can be treated as a section of certain tensor field of a tensor bundle generated by \mathbf{E} . Similarly, we can define the complex structure on \mathbf{E} . $\Lambda^*T^*\mathbf{G}$ are examples of good bundles. Hence, a metric h on $T\mathbf{G}$ defines the Riemannian structure on \mathbf{G} . (\mathbf{G}, h) is called a Riemannian orbifold. Similarly, we can define orbifolds with symplectic forms, almost complex structures and etc.

The integration on \mathbf{G} is not defined on G_0 but on $|\mathbf{G}|$. This is explained in [ALR].

Let $\mathbf{E} \rightarrow \mathbf{G}$ be a bundle with a metric over a Riemannian orbifold. We can define the norms on sections of \mathbf{E} to obtain the Sobolev spaces $W^{k,p}(\mathbf{E})$, etc. They are Banach spaces.

We consider a special case. Let $\mathbf{L} \rightarrow \mathbf{G}$ be a Hermitian line bundle. Suppose that $\mathbf{L} = (L_0, \sigma)$. Let SL_0 be the circle bundle of L_0 . Then $\mathbf{SL} = (SL_0, \sigma)$ is a circle bundle over \mathbf{G} . We claim that \mathbf{SL} is an S^1 -principle bundle over \mathbf{G} in the following sense. For each $t \in S^1$ there is an automorphism $\phi(t) = (\phi_0(t), \phi_1(t))$ of \mathbf{SL} such that $\phi(s)\phi(t) = \phi(st)$. Therefore S^1 also acts on SL_0 and SL_1 . It is easy to see that

$$\mathbf{SL}/S^1 := (SL_0/S^1, SL_1/S^1) \cong \mathbf{G},$$

and

$$(3.1) \quad \mathbf{L} = (L_0, L_1) \cong (SL_0 \times_{S^1} \mathbb{C}, SL_1 \times_{S^1} \mathbb{C}) =: \mathbf{SL} \times_{S^1} \mathbb{C}.$$

Let \mathbb{DL} be the disk bundle of \mathbf{L} , then

$$(3.2) \quad \mathbb{DL} = \mathbf{SL} \times_{S^1} \mathbb{D}.$$

3.1.3. Orbifold morphisms. One of the essential difference between the orbifold theory and the smooth manifold theory is the treatment of *map or morphism*. This is the place where the groupoid/stacky language developed in the last section is very useful. Historically, a great deal of efforts was put into this issue.

Definition 3.8. *Suppose that \mathbf{H}, \mathbf{G} are orbifold groupoids. An orbimorphism between \mathbf{H}, \mathbf{G} is a triple*

$$\mathbf{H} \xleftarrow{\epsilon} \mathbf{K} \xrightarrow{\phi} \mathbf{G},$$

such that the left arrow is an orbifold equivalence.

For any $x \in H_0$, we can invert ϵ locally to obtain a map $U_x \rightarrow U_{\phi\epsilon(x)}$ and a homomorphism $H_x \rightarrow G_{\phi\epsilon(x)}$. We call the above orbifold morphism *representable* if the homomorphism $H_x \rightarrow G_{\phi\epsilon(x)}$ is injective. Next we consider notions of equivalence between morphisms.

- If there exists a natural transformation between $\phi, \phi' : \mathbf{K} \rightarrow \mathbf{G}$ we consider $\mathbf{H} \xleftarrow{\epsilon} \mathbf{K} \xrightarrow{\phi'} \mathbf{G}$ to be equivalent to $\mathbf{H} \xleftarrow{\epsilon} \mathbf{K} \xrightarrow{\phi} \mathbf{G}$.
- If $\delta : \mathbf{K}' \rightarrow \mathbf{K}$ is an orbifold equivalence, $\mathbf{H} \xleftarrow{\epsilon\delta} \mathbf{K}' \xrightarrow{\phi\delta} \mathbf{G}$ is considered to be equivalent to $\mathbf{H} \xleftarrow{\epsilon} \mathbf{K} \xrightarrow{\phi} \mathbf{G}$.

Let \mathcal{R} be the minimal equivalence relation among orbimorphisms generated by the two relations above.

Definition 3.9. *Two orbimorphisms are said to be equivalent if they belong to the same \mathcal{R} -equivalence class.*

The equivalence class of orbimorphisms is independent of orbifold Morita equivalence.

3.1.4. *Chen-Ruan cohomology.* A key concept in the orbifold theory is the Chen-Ruan cohomology. Suppose that \mathbf{G} is an orbifold groupoid. Consider

$$\mathcal{S}_{\mathbf{G}} = \{g \in G_1, s(g) = t(g)\}.$$

Intuitively, an element of $\mathcal{S}_{\mathbf{G}}$ can be viewed as a constant loop. \mathbf{G} acts naturally on $\mathcal{S}_{\mathbf{G}}$ and endow an orbifold groupoid structure with the space of object $\mathcal{S}_{\mathbf{G}}$. We denote such an orbifold groupoid $\wedge \mathbf{G}$. and refer it as an *inertia orbifold*. $\wedge \mathbf{G}$ is an extremely important object and often referred as the *inertia orbifold of \mathbf{G}* .

Recall that, as a set,

$$|\wedge \mathbf{G}| = \{(x, (g)_{G_x}); x \in |\mathbf{G}|, g \in G_x\}.$$

Suppose that p, q are in the same orbifold chart U_x/G_x . Let \tilde{p}, \tilde{q} be a preimage of p, q . Then, $G_p = G_{\tilde{p}}, G_q = G_{\tilde{q}}$ and both of them are subgroup of G_x . We call that $(g_1)_{G_p} \cong (g_2)_{G_q}$ if $g_1 = hg_2h^{-1}$ for some element $h \in G_x$. For two arbitrary points $p, q \in X$, we call $(g)_{G_p} \cong (g')_{G_q}$ if there is a sequence $(p_0, (g_0)_{G_{p_0}}), \dots, (p_k, (g_k)_{G_{p_k}})$ such that $(p_0, (g_0)_{G_{p_0}}) = (p, (g)_{G_p})$, $(p_k, (g_k)_{G_{p_k}}) = (q, (g')_{G_q})$ and p_i, p_{i+1} is in the same orbifold chart and $(g_i)_{G_{p_i}} \cong (g_{i+1})_{G_{p_{i+1}}}$. This defines an equivalence relation on $(g)_{G_p}$.

Let $T_{\mathbf{G}}$ be the set of equivalence classes of conjugacy classes. To abuse the notation, we often use (g) to denote the equivalence class which $(g)_{G_q}$ belongs to. Let $\mathbf{G}_{(g)}$ be the corresponding component.

Then,

$$\wedge \mathbf{G} = \bigsqcup_{(g) \in T_{\mathbf{G}}} \mathbf{G}_{(g)}.$$

Definition 3.10. We call $\mathbf{G}_{(g)}$ for $g \neq 1$ a *twisted sector* and $\mathbf{G}_{(1)} = \mathbf{G}$ the *nontwisted sector*.

Suppose that \mathbf{G} has an almost complex structure. Let $g \in \mathcal{S}_{\mathbf{G}}$ and $p = s(g) = t(g)$. Then, the local group G_p acts on $T_p G_0$ and induce a representation $\rho_p : G_p \rightarrow GL(n, \mathbb{C})$ (here $n = \dim_{\mathbb{C}} G_0$). $g \in G_p$ has finite order. We can write $\rho_p(g)$ as a diagonal matrix

$$\text{diag}(e^{2\pi i m_{1,g}/m_g}, \dots, e^{2\pi i m_{n,g}/m_g}),$$

where m_g is the order of $\rho_p(g)$, and $0 \leq m_{i,g} < m_g$. This matrix depends only on the conjugacy class $(g)_{G_p}$ of g in G_p . We define a function $\iota : |\wedge \mathbf{G}| \rightarrow \mathbb{Q}$ by

$$(3.3) \quad \iota(p, (g)_{G_p}) = \sum_{i=1}^n \frac{m_{i,g}}{m_g}.$$

It is easy to show that ι is locally constant and hence constant on each component.

Definition 3.11. *We define Chen-Ruan cohomology groups $H_{CR}^d(\mathbf{G})$ of \mathbf{G} by*

$$H_{CR}^d(\mathbf{G}) = \bigoplus_{(g) \in T_{\mathbf{G}}} H^d(\mathbf{G}_{(g)})[-2\iota_{(g)}] = \bigoplus_{(g) \in T} H^{d-2\iota_{(g)}}(\mathbf{G}_{(g)}).$$

Here each $H^*(\mathbf{G}_{(g)})$ is the deRham cohomology of rational coefficient \mathbb{Q} . Note that, in general, Chen-Ruan cohomology groups are rationally graded.

Recall that there is a diffeomorphism $I : \mathbf{G}_{(g)} \rightarrow \mathbf{G}_{(g^{-1})}$, which is an involution of $\wedge \mathbf{G}$ as an orbifold.

Suppose that $|\mathbf{G}|$ is a compact, oriented space. For any $0 \leq d \leq 2n$, the pairing

$$\langle \rangle : H_{CR}^d(\mathbf{G}) \times H_{CR}^{2n-d}(\mathbf{G}) \rightarrow \mathbb{Q}$$

defined by the direct sum of

$$\langle \rangle^{(g)} : H^{d-2\iota_{(g)}}(\mathbf{G}_{(g)}) \times H^{2n-d-2\iota_{(g^{-1})}}(\mathbf{G}_{(g^{-1})}) \rightarrow \mathbb{Q}$$

where

$$\langle \alpha, \beta \rangle^{(g)} = \int_{\mathbf{G}_{(g)}} \alpha \wedge I^*(\beta)$$

is nondegenerate.

3.2. Moduli space of stable orbifold morphisms. After the preparation from last subsection, we can introduce the orbifold Gromov-Witten theory along the line of the ordinary Gromov-Witten theory.

3.2.1. Orbifold stable maps. We start from the notation of an *orbifold stable map*, a generalization of stable map in the orbifold category. To do so, we fix a symplectic orbifold groupoid (\mathbf{G}, ω) and equip it with a tamed almost complex structure J .

Definition 3.12. *A stable orbifold morphism or map $f : \mathbf{C} \leftarrow \mathbf{C}' \rightarrow \mathbf{G}$ is a representable, holomorphic orbifold morphism from an orbi-curve \mathbf{C} (possibly nodal) with a finite automorphism. The equivalence relation of stable orbifold morphism is that of orbifold morphism described in previous subsection. An automorphism of f is a \mathcal{R} -equivalence to itself. We define $\overline{\mathcal{M}}_{g,m,A}(\mathbf{G})$ to be the moduli space of the equivalence class of stable orbifold morphism of genus g , m -marked points and degree $A \in H_2(|\mathbf{G}|, \mathbb{Z})$.*

For each marked point x_i , there is an evaluation map

$$ev_i : \overline{\mathcal{M}}_{g,m,A}(\mathbf{G}) \rightarrow \wedge \mathbf{G}.$$

We can use the decomposition of $\wedge \mathbf{G}$ to decompose $\overline{\mathcal{M}}_{g,m,A}(\mathbf{G})$ into components:

$$\overline{\mathcal{M}}_{g,m,A}(\mathbf{G}) = \bigsqcup_{(g_i) \in T_{\mathbf{G}}} \overline{\mathcal{M}}_{g,m,A}(\mathbf{G})((g_1), \dots, (g_m)),$$

where $\overline{\mathcal{M}}_{g,m,A}(\mathbf{G})((g_1), \dots, (g_m))$ is the component being mapped into $\mathbf{G}_{(g_i)}$ under ev_i . For simplicity, we set $(\mathbf{g}) = ((g_1), \dots, (g_m))$ and denote the component by $\overline{\mathcal{M}}_{g,(\mathbf{g}),A}(\mathbf{G})$.

3.2.2. Dual graphs in orbifold setting. The notion of dual graphs (for example, cf. §2.2.2) generalizes to the orbifold setting. Let Γ be a dual graph of a stable map. In the orbifold setting, we assign an additional *orbifold decoration* (g) at each tail and each half edge with the balanced condition that if an edge consists of two half edges τ_+, τ_- with the decoration $(g_+), (g_-)$, we require $g_+ = g_-^{-1}$. Furthermore, the contraction of edge defines a partial order $\Gamma \geq \Gamma'$ if Γ is obtained a sequence of contractions from Γ' . Define \mathcal{M}_{Γ} to be the set of orbifold stable morphisms whose combinatorial type is Γ . Let $\overline{\mathcal{M}}_{\Gamma} = \bigsqcup_{\Gamma' \leq \Gamma} \mathcal{M}_{\Gamma'}$. Then, we obtain a stratification

$$\overline{\mathcal{M}}_{g,(\mathbf{g}),A}(\mathbf{G}) = \bigsqcup_{\Gamma \leq \Gamma_{g,(\mathbf{g}),A}} \mathcal{M}_{\Gamma},$$

where $\Gamma_{g,(\mathbf{g}),A}$ is the dual graph with one vertex and orbifold decorations $(\mathbf{g}) = ((g_1), \dots, (g_m))$.

3.2.3. Orbifold Gromov-Witten invariants. In [CR3], a virtual cycle was constructed for $\overline{\mathcal{M}}_{g,(\mathbf{g}),A}(\mathbf{G})$ with virtual dimension

$$\text{vir dim}_{\mathbb{C}} \overline{\mathcal{M}}_{g,(\mathbf{g}),A}(\mathbf{G}) = c_1(A) + (3-n)(g-1) + m - \iota_{\mathbf{g}},$$

where $\iota_{\mathbf{g}} = \sum \iota_{g_i}$. Let $\alpha_i \in H^*(\mathbf{G}_{(g_i)})$. The orbifold Gromov-Witten theory is defined to be

$$\langle \tau_{l_1}(\alpha_1), \dots, \tau_{l_m}(\alpha_m) \rangle_{g,\mathbf{g},A} = \int_{[\overline{\mathcal{M}}_{g,(\mathbf{g}),A}(\mathbf{G})]^{vir}} \prod_i ev_i^*(\alpha_i) \psi_i^{l_i}.$$

We can use the genus zero invariants to define a quantum product. Let

$$\langle \alpha_1, \dots, \alpha_m \rangle_{g,\mathbf{g},A} = \langle \tau_0(\alpha_1), \dots, \tau_0(\alpha_m) \rangle_{g,\mathbf{g},A}.$$

Then, we define the quantum product $\alpha_1 * \alpha_2$ by the formula

$$\langle \alpha_1 * \alpha_2, \gamma \rangle = \sum_A \langle \alpha_1, \alpha_2, \gamma \rangle_{0,A} q^A.$$

The Chen-Ruan product $\alpha_1 \cup_{CR} \alpha_2$ is defined using above formula with $\langle \alpha_1, \alpha_2, \gamma \rangle_{0,0}$ on the right hand side of equation.

4. MODULI SPACE OF RELATIVE ORBIFOLD STABLE MAPS

After reviewing the relative and orbifold stable maps, it should be clear now how to merge them to set up the notion of relative orbifold stable maps.

In this section, we simultaneously consider (i) the moduli spaces of relative stable maps to (G, Z) , (ii) the moduli space of stable maps to the degenerated orbifold $G^- \wedge_Z G^+$.

4.1. Orbifold relative pairs and degenerations. Suppose that G is a symplectic orbifold groupoid. Let $Z_0 \subset G_0$ be a submanifold invariant under the action of G_1 . Then, we can define an orbifold groupoid Z with Z_0 and $Z_1 = \{g \in G_1, t(g), s(g) \in Z_0\}$. $Z \subset G$ is called a *subgroupoid*.

We further assume that Z is a symplectic divisor. By doing this, again we standardize the neighborhood of Z in G as (2.2). We assume that there is a neighborhood U of Z such that $U \cong \mathbb{D}N$, where N is a line bundle over Z (cf. (3.2)). Let $Y = SN$. Then the construction of §2.1 generalizes to the orbifold setting word by word. In particular, let $Q = \mathbb{P}(N \oplus \mathbb{C})$ be the projection of N , or $Q = Y \times_{S^1} \mathbb{CP}^1$.

Now suppose we have two pairs (G^\pm, Z) . Let N^\pm be the normal bundles of Z in G^\pm . We say that G^\pm intersect at Z normal crossingly if N^+ and N^- are inverse to each other. We define a degenerated groupoid $G^+ \wedge_Z G^-$ by

$$(G^+ \wedge_Z G^-)_0 = G_0^+ \wedge_{Z_0} G_0^-, \quad (G^+ \wedge_Z G^-)_1 = G_1^+ \wedge_{Z_1} G_1^-.$$

Since

$$(4.1) \quad N^+ \otimes N^- \cong Z \times \mathbb{C},$$

Proposition 2.1 can be extended to the orbifold setting

Proposition 4.1. *Let $G = G^+ \wedge_Z G^-$ be a degenerated groupoid. There is a smooth family of symplectic groupoid $\pi : (\mathcal{D}, \omega) \rightarrow \mathfrak{D}$ such that $\pi^{-1}(0) = G$.*

4.1.1. Examples. For a pair (G, Z) , we constructed Q, Q_m, G_m in the same way.

Another important example is an orbifold curve with balanced nodal points (cf. Example 3.1). By the definition, it is clear that such a curve is a degenerated orbifold. Let $C = C^+ \wedge C^-$ be such a curve and its nodal point is $y = y^+ = y^-$. Suppose the orbifold structure is marked

by \mathbb{Z}_r and denote the curve by C_r . We apply Proposition 4.1 to C_r and construct families

$$\pi_r : \mathcal{D}_r \rightarrow \mathfrak{D}_r.$$

Then we have a simple fact which is crucial for the construction of the gluing bundle in gluing theory (cf. §5.3.2).

Lemma 4.2. $\mathfrak{D}_1 = \mathfrak{D}_r / \mathbb{Z}_r$.

Proof. Let \mathbb{C}^\pm be the normal line of y^\pm in C^\pm (forgetting the orbifold structure, i.e, taking $r = 1$). \mathfrak{D}_1 is the disk of $\mathbb{C}^+ \otimes \mathbb{C}^-$.

For C_r , \mathbb{C}^\pm is identified with $\tilde{\mathbb{C}}^\pm / \mathbb{Z}_r$. \mathfrak{D}_r is the disk of

$$\tilde{\mathbb{C}}^+ \otimes_{\mathbb{Z}_r} \tilde{\mathbb{C}}^- = \tilde{\mathbb{C}}^+ \otimes \tilde{\mathbb{C}}^-.$$

Here, we use the fact that \mathbb{Z}_r acts trivially on the space of tensor product. Hence, it is easy to see that \mathfrak{D}_r is an r -branch cover of \mathfrak{D}_1 . q.e.d.

By the construction of the family, we note that $\pi_r^{-1}(t) = \pi_1^{-1}(t^r)$.

4.2. Fractional contact order. Recall the contact order in smooth case. Consider a non-constant orbifold curve $f : \Sigma \rightarrow X$. and suppose that $f^{-1}(Z)$ consists of isolated points on Σ :

$$f^{-1}(Z) = \{y_1, \dots, y_k\}.$$

Formally, the intersection of the curve with Z is expressed as $f(\Sigma) \cap Z = \sum_{i=1}^k \ell_i f(y_i)$. ℓ_i is called the *contact order* of the curve with Z at $f(x_i)$. Complex analytically, it can be described as follows. Locally, we express

$$f : \mathbb{D}_{x_i} \rightarrow V \times \mathbb{C}, \quad f(w) = (f_1(w), w^{\ell_i} + O(w^{\ell_i+1})),$$

where $V \times \mathbb{C}$ is a local neighborhood of $z = f(x_i)$ such that V is a neighborhood in Z and \mathbb{C} is the fiber of normal bundle.

One can also compute it topologically as a degree. Let Θ be a Thom form of the normal bundle that supported in a small neighborhood of Z . Then the restriction of Θ on fiber \mathbb{C} is a 2-form with $\int \Theta = 1$. $f^*\Theta$ is supported in a small neighborhood of x_i and $\ell_i = \int_{\mathbb{D}_{x_i}} f^*\Theta$.

Now suppose that f is an orbifold morphism $\mathbf{f} = (f_0, f_1)$ and $f = |\mathbf{f}|$. Let $z = f(y_i)$. Locally

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{f_0} & V \times \mathbb{C} \\ \downarrow / \mathbb{Z}_r & & \downarrow / G_z \\ \mathbb{D} & \xrightarrow{f} & (V \times \mathbb{C}) / G_z \end{array}$$

and f_1 yields an (injective) morphism $\mathbb{Z}_r \rightarrow G_z$. As a map,

$$(4.2) \quad f_0 : \mathbb{D} \rightarrow V \times \mathbb{C}, \quad f_0(w) = (f^1(w), w^d + O(w^{d+1})).$$

Definition 4.1. *For the map given by (4.3), we define the fractional contact order at $z = f(0)$ to be $\ell = d/r$.*

We have the following fact.

Lemma 4.3. *Suppose that the image of f intersects with $|Z|$ at finitely many points z_1, \dots, z_k and their preimages are y_1, \dots, y_k . Then*

$$f_*[C] \cap [Z] = f^*[Z] \cap [C] = \sum_i \ell_{y_i}.$$

Proof. Let Θ be the Thom form of normal bundle $N_{Z|G}$. Namely, Θ is the volume form on \mathbb{C} of volume one. Then

$$f^*[Z] \cap [C] = \int_C^{orb} f^*\Theta$$

can be expressed as a sum of the local contribution of y_i 's. Furthermore, the local contribution at x_i is

$$\int_{\mathbb{D}/\mathbb{Z}_r} f^*\Theta = \frac{1}{r} \int_{\mathbb{D}} \tilde{f}^*\Theta = \frac{d}{r}.$$

q.e.d.

4.3. Stable relative orbifold morphisms to (G, Z) . The definition is similar to that of relative stable map. Suppose that C is an orbicurve, (possibly smooth). We divide its components, marked points, nodal points into the absolute and relative types.

Definition 4.2. *A stable relative orbifold holomorphic morphism or map f is a triple*

$$C \xleftarrow{\epsilon} C' \xrightarrow{\phi} G_m$$

such that ϵ is a holomorphic equivalence and ϕ is a holomorphic morphism with the properties. Here C and C' are equivalence orbifold structures for C . Furthermore, we require that

- (1) *The absolute components are mapped into G and the relative components are mapped into Q_m .*
- (2) *The preimage of $f^{-1}(|\cup_{i=0}^m Z_i|)$ consists of all the relative marked points and nodes.*
- (3) *The relative marked points are mapped into $Z_{m,0}$ and the sum of intersection multiplicities equals to $Z \cdot A$.*

- (4) *The relative nodes are mapped into $\text{Sing}\mathbf{G}_m$ satisfying balanced condition that the two branches at the node are mapped to different irreducible components of \mathbf{G}_m and the contact orders to $Z_{i,\infty} = Z_{i-1,0}$ are equal.*
- (4) *The automorphism group is finite.*

The equivalence relation is that of \mathcal{R} -equivalence and the automorphism of \mathbf{Q}_m (see Definition 3.9). An automorphism is a self equivalence.

Let $\overline{\mathcal{M}}_{g,m,A,T_k}(\mathbf{G}, \mathbf{Z})$ be the space of stable relative orbifold morphism with genus g , fundamental class A , number of absolute marked point m , relative marked points with the contact orders prescribed by T_k .

For each marked point, we have an evaluation map. If the marked point y_i is absolute, we have

$$ev_i : \overline{\mathcal{M}}_{g,m,A,T_k}(\mathbf{G}, \mathbf{Z}) \rightarrow \wedge \mathbf{G}.$$

If the marked point x_i is relative, we have a *relative* evaluation map

$$ev_j^r : \overline{\mathcal{M}}_{g,m,A,T_k}(\mathbf{G}, \mathbf{Z}) \rightarrow \wedge \mathbf{Z}.$$

Let $(\mathbf{g}) = \{(g_1), \dots, (g_m)\}$, $(\mathbf{h}) = \{(h_1), \dots, (h_k)\}$. We have the decomposition

$$\overline{\mathcal{M}}_{g,m,A,T_k}(\mathbf{G}, \mathbf{Z}) = \bigsqcup_{(\mathbf{g}), (\mathbf{h})} \overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z}),$$

by specifying the monodromies at marked points.

Next, we generalize the dual graph to the relative orbifold setting. For each relative orbifold stable morphism, we assign a (*connected*) *relative orbifold graph* Γ called *type* consisting of the following data:

- (1) a vertex decorated by $A \in H_2(|\mathbf{G}|; \mathbb{Z})$, genus g , a level i for each component,
- (2) an absolute tail decorated by a conjugacy class (g) of \mathbf{G} for each absolute marked point,
- (3) a relative tail decorated by its contact order and conjugacy class (h) of \mathbf{Z} for each relative marked point.
- (4) an absolute edge with orbifold decoration $(g), (g^{-1})$ of \mathbf{G} on the half edges for each absolute node.
- (5) a relative edge decorated by the contact order and orbifold decoration $(h), (h^{-1})$ of \mathbf{Z} on the half edges for each relative node.

Furthermore, the sum of contact orders of relative tails equals to $\mathbf{Z} \cdot A$ and the levels of two adjacent vertices are same or different by 1.

Let $T_k = \{\ell_1, \dots, \ell_k\}$ be a partition of $\mathbf{Z} \cdot A$ and $\mathcal{S}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}$ be the set of relative graphs with genus g , fundamental class A , m -absolute tails decorated by the conjugacy classes (\mathbf{g}) , k -relative tails decorated

by the partition T_k , conjugacy classes (\mathbf{h}) . As in the smooth case, the contraction induces a partial order on $\mathcal{S}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}$. There is a unique maximal graph $\Gamma_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}$ with one vertex. For each Γ , let $\mathcal{M}_\Gamma(\mathbf{G}, \mathbf{Z})$ be the space of orbifold relative morphisms of Γ -type. Then

$$\begin{aligned}\mathcal{M}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z}) &= \mathcal{M}_{\Gamma_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}}(\mathbf{G}, \mathbf{Z}), \\ \overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z}) &= \bigsqcup_{\Gamma \in \mathcal{S}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}} \mathcal{M}_\Gamma(\mathbf{G}, \mathbf{Z}).\end{aligned}$$

Lemma 4.4. *The virtual dimension of $\mathcal{M}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z})$ is given by the formula*

$$(4.3) \quad c_1(A) + (3-n)(g-1) + m + k - \iota_{(\mathbf{g})} - \iota_{(\mathbf{h})} - \sum_i [\ell_i],$$

where $[\ell_i]$ is the biggest integer less than ℓ_i .

Proof. First of all, we ignore the relative data and consider the moduli space $\mathcal{M}_{g,(\mathbf{g} \cup \mathbf{h}),A}(\mathbf{G})$. Then its virtual dimension is

$$\tilde{d} = c_1(A) + (3-n)(g-1) + m + k - \iota_{(\mathbf{g})} - \iota_{(\mathbf{h})}.$$

as the index of the elliptic complex

$$\bar{\partial} : \Omega^0(\mathbf{f}^* \mathbf{T}\mathbf{G}) \rightarrow \Omega^{0,1}(\mathbf{f}^* \mathbf{T}\mathbf{G})$$

for any $\mathbf{f} \in \mathcal{M}_{g,(\mathbf{g} \cup \mathbf{h}),A}(\mathbf{G})$. The index is equivalent to the index of same complex for the desingularization $|\mathbf{f}^* \mathbf{T}\mathbf{G}|$ (see Proposition 4.2.1 [CR1]). The later has the desired expression.

Now we consider the relative data. That is, we are interested in the subspace of the sections s of $\mathbf{f}^* \mathbf{T}\mathbf{G}$ that has contact order ℓ_i at relative marking y_i . We claim that the corresponding section \tilde{s} of $|\mathbf{f}^* \mathbf{T}\mathbf{G}|$ has order $[\ell_i]$ and the lemma follows. We verify this. Suppose that locally

$$\mathbf{f}^* \mathbf{T}\mathbf{G} \cong (\mathbb{D} \times \mathbb{C}^n) / \mathbf{Z}_r,$$

where the last factor represents the normal bundle. Let $\zeta \in \mathbb{Z}_m$ be the generator. Suppose that

$$\zeta(z, v_1, \dots, v_{n-1}, v_n) = (\zeta z, \zeta^{t_1} v_1, \dots, \zeta^{t_n} v_n).$$

s has the local form $(s_1(z), \dots, s_n(z))$ with the property that $s_i(\zeta z) = \zeta^{t_i} s_i(z)$. Let k be the lowest degree of s_n . Then, $k = t_n + pr$ for some integer p . Let $u = z^r$ be the coordinate of D/\mathbb{Z}_r . Then \tilde{s} has the local form $\tilde{s}_n(u) = z^{-t_n} s_n(z^r)$. Its lowest degree is $p = [k/m]$. q.e.d.

4.4. **Stable morphisms to $G^+ \wedge_Z G^-$.** We define

$$(G^+ \wedge_Z G^-)_m = G^+ \wedge_{Z_{0,0}} Q_m \wedge_{Z_{m,\infty}} G^-.$$

Intuitively, a stable morphism is a morphism from orbifold curve C to $(G^+ \wedge_Z G^-)_m$. The definition is identical to that of stable relative orbifold morphism. But here we do not need the relative marked points. Then, we can copy the definitions from last section word by word to define *stable orbifold morphism* to $G^+ \wedge_Z G^-$ and the moduli space $\overline{\mathcal{M}}_{g,(\mathbf{g}),A}(G^+ \wedge_Z G^-)$. The definition of dual graphs and its stratification are identical as well.

Recall that we have a degeneration family $\pi : \mathcal{D} \rightarrow \mathfrak{D}$ for G (cf. Proposition 4.1). Let $G_t = \pi^{-1}(t)$. Consider a natural (topological) map $\phi_t : |G_t| \rightarrow |G|$ which induces a map $\phi_{t,*} : H_2(|G_t|) \rightarrow H_2(|G|)$. In fact, $\phi_{t,*}$ is independent of t and we denote it by ϕ_* . ϕ_* may map different homology classes to the same one. Intuitively, holomorphic maps of different fundamental classes in G_t may converge to the holomorphic maps of the same fundamental class in G . Let $[A] = \phi_*^{-1}(A)$ and

$$\overline{\mathcal{M}}_{g,(\mathbf{g}),[A]}(G_t) = \bigsqcup_{B \in [A]} \overline{\mathcal{M}}_{g,(\mathbf{g}),B}(G_t).$$

We define the moduli space of the family

$$(4.4) \quad \overline{\mathcal{M}}_{g,(\mathbf{g}),[A]}(\mathcal{D}) = \overline{\mathcal{M}}_{g,(\mathbf{g}),\phi_*([A])}(G) \times \{0\} \cup \bigsqcup_{t \in \mathfrak{D}^*} \overline{\mathcal{M}}_{g,(\mathbf{g}),[A]}(G_t) \times \{t\}$$

Then we also have a natural projection $\pi : \overline{\mathcal{M}}_{g,(\mathbf{g}),[A]}(\mathcal{D}) \rightarrow \mathfrak{D}$. This can be thought as a degeneration on the moduli space level.

4.5. **Compactness.** In this subsection, we establish the compactness of moduli spaces of orbifold relative stable maps. The smooth case was first established by Li-Ruan (§3, [LR]), where they adapted the cylinder end model and introduced the rubber components. In the orbifold case, we will first apply the argument from smooth case to obtain a convergence of underline maps. The remaining issue is to put appropriate orbifold structure on the limit nodal curve and lift the limit map to an orbifold morphism.

Theorem 4.5. $\overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(G, Z)$ is compact.

Proof. Suppose we have a sequence of orbifold morphisms in the moduli space, denoted by

$$f_i = \{C_i \leftarrow C'_i \xrightarrow{\phi_i} G\}.$$

In the proof, we write this as $f_i : C_i \xrightarrow{\sim} G$. Here for simplicity, we only consider the case that the target space is G other than G_m . Also, for

simplicity, we assume that $|\mathbf{C}_i|$ is smooth. The proof for general cases is essentially encoded in the proof itself. By equipping both \mathbf{C}_i and \mathbf{G} a metric, we can define the gradient $|\nabla \mathbf{f}_i|$. It is convenient to use the cylindric metric at the end of \mathbf{C}_i at relative marked points and that of \mathbf{G} at \mathbf{Z} .

The proof of the theorem consists of §4.5.1-§4.5.4.

4.5.1. *Convergence of underline map $|\mathbf{f}_i|$.* Consider $f_i = |\mathbf{f}_i| : |\mathbf{C}_i| \rightarrow |\mathbf{G}|$. Since an orbifold is locally the quotient of a smooth manifold by a finite group, we can attempt to work on a local lift of f_i and apply the technique from smooth case. It was observed already in [CR2] that the above strategy indeed works. By applying the argument from [LR], we obtain a subsequence converging to a relative stable map $f_\infty : \Sigma \rightarrow |\mathbf{G}_m|$, where Σ is a nodal curve and f_∞ is locally lift to a holomorphic map. A subtle issue is to *endow an orbifold structure naturally on Σ and lift f_∞ to an orbifold morphism*.

Instead of copying the proof from [LR](cf. §3 [LR]), we describe the convergence process of f_i and omit the details. It consists of 2 steps.

Let S be set of special points: marked points and nodal points. Set

$$S^{(n)} = \bigcup_{x \in S} \mathbb{D}_{1/n}(x).$$

Here we take the flat metric at a neighborhood of the point, which is naturally identified with a cylinder end. Let $\Sigma_{reg} = \Sigma \setminus S$ and $K^{(n)} = \Sigma \setminus S^{(n)}$. Clearly, $K^{(n)}$ exhausts Σ_{reg} .

Step 1, Convergence on Σ_{reg} . For each $K = K^{(n)}$, (in the following, we omit the index (n) if no confusion may be caused), there are

- an embedding $\lambda_i : K \rightarrow |\mathbf{C}_i|$;
- a small constant $r_i > 0$ and an embedding $\mu_i : |\mathbf{G}_{m,r_i}| \rightarrow |\mathbf{G}^*|$, where \mathbf{G}_{m,r_i} is the complement of all disk bundle $\mathbb{D}_{r_i} \mathbf{N}_j^\pm$ over all \mathbf{Z}_j .

such that for the map \tilde{f}_i defined by the diagram

$$(4.5) \quad \begin{array}{ccc} K & \xrightarrow{\tilde{f}_i} & |\mathbf{G}_{m,r_i}| \\ \lambda_i \downarrow & & \mu_i \downarrow \\ |\mathbf{C}_i| & \xrightarrow{f_i} & |\mathbf{G}^*| \end{array}$$

have the properties:

- (1) $|\nabla \tilde{f}_i|$ is uniformly bounded over K ;
- (2) the pull-back of the complex structure from \mathbf{C}_i converges to that of K ;

$$(3) \lim \tilde{f}_i = f_\infty.$$

Here G^* is the groupoid for the cylinder end model. We remark that at cylinder end of either G or C_i , we choose the groupoid structure to be

$$(4.6) \quad (-\infty, \ln \epsilon) \times Y.$$

This allows us to write μ_i as an embedding $\mu_i : G_{m,r_i} \rightarrow G^*$.

Step 2, Converges at a point $x \in S$. Since f_∞ is defined on Σ_{reg} , it can be extended over on Σ by the standard removable singularity argument and no energy lost arguments for holomorphic maps.

4.5.2. Orbifold structures on Σ_{reg} . In fact, the argument in this subsection also works for absolute marked points. Again we first consider $K = K^{(n)}$. Let $K_i = \lambda_i(K)$. By the refinement if necessary, the orbifold morphism $f_i : C_i \xrightarrow{\sim} G$ can be restrict on K_i and induces a morphism, still denoted by f_i , from $K_i \xrightarrow{\sim} G$. Be precise, we have a groupoid morphism $\phi_i : K'_i \rightarrow G$, where K'_i is the groupoid structure induced from C'_i . Via λ_i , we may assume that K'_i is an orbifold structure on K . This make K to be an orbifold Riemannian surface. Let K be any arbitrary fixed orbifold structure for this curve. Then we just have

$$f'_i : K \leftarrow K'_i \xrightarrow{\phi_i} G.$$

The diagram, in terms of groupoids, is

$$(4.7) \quad \begin{array}{ccc} K'_i & \xrightarrow{\phi_i} & G \\ \downarrow & & \uparrow \mu_i \\ K & \xrightarrow{\sim} & G_{m,i}. \end{array}$$

In order to claim that we do have the orbifold morphism. We should be able to reverse μ_i : this can be done because of (4.6). Therefore, we have a morphism

$$(4.8) \quad \tilde{f}'_i : K \leftarrow K'_i \xrightarrow{\tilde{\phi}_i} G_m,$$

where $\tilde{\phi}_i$ given by the composition

$$\tilde{\phi}_i : K'_i \xrightarrow{\phi_i} G \xrightarrow{\mu_i^{-1}} G_{m,i} \hookrightarrow G_m.$$

To obtain the limiting orbifold structure, we need to take limit on the groupoid level. The main problem is that K'_i does not have any limit in general. We have to change f'_i in its equivalence class to achieve the same domain topologically. This is stated in the following lemma.

Lemma 4.6. *If $\tilde{f}'_i = |\tilde{f}'_i| = \tilde{f}_i$ converges to f_∞ and $|\nabla \tilde{f}'_i| < N$ is uniformly bounded, then there exists an orbifold structure K^o that Morita equivalent to K and a groupoid morphism*

$$\tilde{f}_i : K^o \rightarrow G_m$$

such that \tilde{f}_i and \tilde{f}'_i are \mathcal{R} -equivalence. Here K^o is independent of i .

Proof. For the sake of notation, we write f' for \tilde{f}'_i in the proof. K'_i is denoted by $K' = (K'_0, K'_1)$.

Let $x \in K'_0$ and $z = f'_0(x) \in G_0$. For simplicity, we assume that $x \in K_0$ of K . Suppose that V_0 is a connected component that contains a ball $B_r(z)$ of z . Let $D_\epsilon(x)$ be a disk of x in K_0 with $\epsilon < r/N$. Let

$$K''_0 = K_0 \sqcup \mathbb{D}_\epsilon(x)$$

and define K''_1 accordingly such that $K'' = (K''_0, K''_1)$ is equivalent to K' . We assert that

Claim. f' can be extended over $f : K'' \rightarrow G$.

Proof of the claim. The point is to construct $f_0 : \mathbb{D}_\epsilon(x) \rightarrow V_0$ naturally. Note that by the control of radius, $f_0(\mathbb{D}_\epsilon(x)) \subset V_0$.

We prove the extension property along lines. Then the assertion follows by the simply-connectness of $\mathbb{D}_\epsilon(x)$. For simplicity, Let $c : [0, 1] \rightarrow \mathbb{D}_\epsilon(x)$ is any curve with $c(0) = x$. For any $y = c(h_0)$ there exists a small disk $\mathbb{D}_\delta(y') \subset K'_0$ such that there is an arrow from α_y that from y to y' . By the local diffeomorphism of s and t , there is an interval $I = (h_0 - \delta_0, h_0 + \delta_0)$ and a path $\alpha(I) \subset K'_1$ with $\alpha(h_0) = \alpha_y$ such that $\alpha(h) \cdot c(h) \in D_\delta(y')$. Hence, we have finite y_i in the path, intervals $I_i = (h_i - \delta_i, h_i + \delta_i)$ and paths $\alpha(I_i)$ such that $c(I_i)$ covers the line.

Now we explain how to define f_0 on I_1 . Let $\beta = f'_1(\alpha(h))$, where h is slightly larger than $h_1 - \delta_1$. On the other hand, we have a path $f'_0(s(\alpha(I_1)))$. By the local diffeomorphism property of s and t in G_m there is a path $\beta(I_1)$ of arrows in G_m . Set $f_0(h) = t(\beta(h))$.

By finite steps, we define f_0 along c . It is easy to see that the construction is independent of the choice of $D_\delta(y')$ and the arrow α_y in the construction. Hence we have define $f_0 : D_\epsilon(x) \rightarrow G_{m,0}$.

On the other hand, we can modify the map f'_1 on new arrows in K'' properly. This completes the proof of the claim.

Since $\overline{|K|}$ is compact, we may choose a finite covering $\mathbb{D}_\epsilon(x_i)$ of $|K|$. Let \tilde{K} be the groupoid by adding these charts to K' and we get a morphism f from \tilde{K} to G_m .

Now we can take $K^o \subset \tilde{K}$ by extracting these finite charts $\mathbb{D}_\epsilon(x_i)$. The restriction of f on K^o then solves the problem. q.e.d.

4.5.3. Orbifold structures at punctured disks of points in S . In order to exhaust $K^{(n)}$, $n \rightarrow \infty$, §4.5.2 is not enough. We should uniformly give a groupoid structure at punctured disks at special points. (In fact, the argument in §4.5.2 already works for the neighborhood of absolute markings.)

Consider the covering $K^{(2N)} \cup S^{(N)}$ of Σ_{reg} . We already construct the uniform groupoid structure $K^{(2N)}$ on $K^{(2N)}$ for all f_i . It remains to construct the groupoid structure on punctured $S^{(N)}$ and morphism f_i^* .

Let $\mathbb{D}^*(x) := \mathbb{D}_\epsilon^*(x)$ be a puncture disk at $x \in S$. As (4.6), we suppose that

$$\mathbb{D}^*(x) := (-\infty, \ln \epsilon) \times S,$$

where S is a groupoid structure on S^1 given by a covering of two intervals U^\pm . Then $\mathbb{D}^*(x)$ is given by charts $\tilde{U}^\pm = (-\infty, \ln \epsilon) \times U^\pm$. Again, $f'_i : C'_i \xrightarrow{\sim} G$ induces morphisms

$$\tilde{f}'_i : \mathbb{D}(x) \setminus \mathbb{D}_{\epsilon_i}(x) \xrightarrow{\sim} G_m.$$

By the same argument of Lemma 4.6, we conclude that there are morphisms

$$f_i : \mathbb{D}(x) \setminus \mathbb{D}_{\epsilon_i}(x) \rightarrow G_m.$$

that is \mathcal{R} -equivalent to \tilde{f}'_i .

Combine with §4.5.2, we finish the step 1 in §4.5.1 on the groupoid level. In particular, we have

$$(4.9) \quad f_\infty^* : \mathbb{D}^*(x) \rightarrow G_m.$$

4.5.4. Fill in the orbifold structure at nodes. As we already have (4.9), we claim that it *enforces* a groupoid $\mathbb{D}(x)$ on $\mathbb{D}(x)$ and a morphism

$$f_\infty : \mathbb{D} \rightarrow G_m$$

such that when restricting on $|\mathbb{D}|^*$, f_∞ is equivalent to f_∞^* .

Focus on $\mathbb{D}^*(x)$, we may modify the groupoid structure S of S^1 to be given by the covering

$$\exp\{2\pi i \cdot\} : [0, 1.5] \rightarrow S^1.$$

Suppose $f_\infty(x) = z$ and the local groupoid structure at z is V_z/G_z . f_∞^* maps the arrow to a group element $g \in G_z$. Let $r = |g|$ then it is standard to construct a groupoid morphism

$$f_\infty : \mathbb{D}/\mathbb{Z}_r \rightarrow V_z/G_z$$

that extends f_∞^* .

The balanced condition at nodes follows from the no energy lost argument. We skip it here. For example, readers are referred to [CR2].

5. VIRTUAL FUNDAMENTAL CYCLES

In this section, we construct virtual fundamental cycles for the compactified spaces

$$[\overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G},\mathbf{Z})]^{vir}, \quad [\overline{\mathcal{M}}_{g,(\mathbf{g}),[A]}(\mathcal{D})]^{vir}$$

of expected dimensions.

5.1. The Kuranishi structure. There are several approaches in the literature. Here, we use the approach of Kuranishi structures by Fukaya-Ono [FO]. The most part of construction is almost same as that of Li-Ruan [LR] and we will be sketchy. Let's first recall the definition of Kuranishi structure.

Let X be a compact, metrizable topological space.

Definition 5.1. *Let V be an open subset of X . A Kuranishi or virtual neighborhood of V is a system (U, E, G, s, Ψ) where*

- (1) $\tilde{U} = U/G$ is an orbifold, and $E \rightarrow U$ is a G -equivariant bundle.
- (2) s is a G -equivariant continuous section of E .
- (3) Ψ is a homeomorphism from $s^{-1}(0)$ to V in X .

We call E the obstruction bundle and s the Kuranishi map. We say (U, E, G, s, Ψ) is a Kuranishi neighborhood of a point $p \in X$ if p has a neighborhood V carrying a Kuranishi neighborhood.

Definition 5.2. *Let $(U_i, E_i, G_i, s_i, \Psi_i)$ be a Kuranishi neighborhood of V_i and $f_{21} : V_1 \rightarrow V_2$ be an open embedding. A morphism*

$$\{\phi\} : (U_1, E_1, G_1, s_1, \Psi_1) \rightarrow (U_2, E_2, G_2, s_2, \Psi_2)$$

covering f is a family of open embeddings

$$\phi_{21} : U_1 \rightarrow U_2, \quad \hat{\phi}_{21} : E_1 \rightarrow E_2, \quad \lambda_{21} : G_1 \rightarrow G_2,$$

$$\Phi_{21} : \phi_{21}^* TU_2 / TU_1 \rightarrow \phi_{21}^* E_2 / E_1$$

(called injections) such that

- (1) $\phi_{21}, \hat{\phi}_{21}$ are λ_{21} -equivariant and commute with bundle projection.
- (2) λ_{21} induces an isomorphism from $\ker(G_1)$ to $\ker(G_2)$, where $\ker(G)$ is the subgroup acting trivially.
- (3) $s_2 \phi_{21} = \hat{\phi}_{21} s_1$ and ϕ_{21} covers $f_{21} : V_1 \rightarrow V_2$; $\Psi_2 \phi_{21} = \Psi_1$
- (4) If $g \phi_{21}(U_1) \cap \phi_{21}(U_1) \neq \emptyset$ for some $g \in G_2$, then g is in the image of λ_{21} .

- (5) G_2 acts on the set $\{\phi_{21}\}$ transitively, where $g(\phi_{21}, \hat{\phi}_{21}, \lambda_{21}) = (g\phi_{21}, g\hat{\phi}_{21}, g\lambda_{21}g^{-1})$.
- (6) Φ_{21} is an G -equivariant bundle isomorphism.

Definition 5.3. A Kuranishi structure of dimension n on X is an open cover \mathcal{V} of X such that

- (1) Each $V \in \mathcal{V}$ has a Kuranishi neighborhood (U, E, G, s, Ψ) such that $\dim U - \dim E = n$.
- (2) If $V_2 \subset V_1$, the inclusion map $i_{12} : V_2 \rightarrow V_1$ is covered by a morphism between their Kuranishi neighborhoods.
- (3) For any $x \in V_1 \cap V_2$, $V_1, V_2 \in \mathcal{V}$, there is a $V_3 \in \mathcal{V}$ such that $x \in V_3 \subset V_1 \cap V_2$.
- (4) The composition of injections is an injection.

Given a Kuranishi structure, Fukaya-Ono [FO] constructed a virtual fundamental cycle whose dimension is given by the index.

In all the known cases, the patching part of construction are same and so is our case. We will not repeat it here. Instead, we will focus the construction of local Kuranishi neighborhood. We will divide it two cases, top stratum and lower stratum. The first case requires a Fredholm analysis while the second case requires additional gluing construction.

5.2. Kuranishi structure at top stratum. Consider a moduli space of relative orbifold stable maps. For simplicity, we assume that there is no absolute marked point and only one relative point with relative monodromy (h) and the contact order is $\ell = k/|h|$.

5.2.1. Weighted Sobolev norms. Let (\mathbb{C}, y) be an orbifold Riemann surface with a relative marked point. It can be thought as an orbifold relative pair. Let \mathbb{C}^* be the cylindric end model for the pair (§2.1.2). We assume that the cylinder end is $(-\infty, 0) \times \mathbb{Y}$ with the standard cylindric metric. (Here $\mathbb{Y} = S^1/\mathbb{Z}_{|h|}$).

In general, Let \mathbb{E} be a vector bundle over \mathbb{C}^* with a metric. Fix a function $\eta(s)$ supported in (T_0, ∞) for some constant T_0 and is 1 when $t \geq T_0 + 1$. This induces a function on supported in $(-\infty, 0) \times \mathbb{Y}$ and hence a function on \mathbb{C}^* .

Let $\alpha > 0$ be a small constant. For a section σ of \mathbb{E} we define the norms

$$\begin{aligned} \|\sigma\|_{p,\alpha} &= \|\sigma\|_{L^p(|\mathbb{C}^*|)} + \|e^{\alpha\eta}\sigma\|_{L^2(|\mathbb{C}^*|)}, \\ \|\sigma\|_{1,p,\alpha} &= \|\sigma\|_{L^{1,p}(|\mathbb{C}^*|)} + \|e^{\alpha\eta}(|\sigma| + |\nabla\sigma|)\|_{L^2(|\mathbb{C}^*|)}, \end{aligned}$$

where

$$\|\sigma\|_{L^p(|C^*|)} = \left(\int_{|C^*|} |\sigma|^p d\mu \right)^{1/p}.$$

Let $L^{p,\alpha}(C^*, E)$ and $W^{1,p,\alpha}(C^*, E)$ be the completion of the spaces of smooth sections of E with respect to these norms.

5.2.2. Weighted Sobolev maps. We follow the set-ups in [LR] and use the orbifolds with cylindric ends.

Let $z \in Z_{(h)}$ and suppose the fiber of the normal bundle over z is \mathbb{C}/G_z (we allow the action to be trivial). A local orbifold map

$$f : \mathbb{D}_\epsilon / \mathbb{Z}_{|h|} \rightarrow \mathbb{C}/G_z$$

is called a (h, ℓ) -relative map if the lifting \tilde{f} of f is $\tilde{f}(w) = aw^k + O(w^{k+1})$. Two local orbifold maps are called equivalent if they match on a small disk. A germ of local map is an equivalence class of local maps. Let $\mathcal{O}_{(h),\ell}$ be the space of germs of (h, ℓ) -relative maps.

Definition 5.4. (1) Let $[f] \in \mathcal{O}_{(h),\ell}$. A map $u : C^* \rightarrow G^*$ is called an $[f]$ -type relative map if there exists a constant T such that the map $u : \mathbb{D}_\epsilon / \mathbb{Z}_{|h|} \rightarrow G$, which is induced by $u : (-\infty, T) \times Y \rightarrow G^*$, yields $[u] = [f]$.

(2) A map $u : C^* \rightarrow G^*$ is called α -exponential decay of $[f]$ -type if there is a $[f]$ -type relative map u' such that $u - u'$ is in $W^{1,p,\alpha}$. We denote the space to be $\mathcal{B}_{[f]}^{1,p,\alpha}$.

(3) let $\mathcal{B}_{h,\ell}^{1,p,\alpha}$ to be the union of $\mathcal{B}_{[f]}^{1,p,\alpha}$, $[f] \in \mathcal{O}_{h,\ell}$.

In (2), u can be expressed as $\exp_{u'} \xi$ for some vector field ξ over u' . Formally we treat ξ as $u - u'$. Hence, in (2), we mean that $\xi \in W^{1,p,\alpha}$.

5.2.3. Orbifold structure on the space of orbifold morphisms. So far, we discuss the space of orbifold morphisms (not necessarily holomorphic) as a set. Similar to the smooth case, its completion with respect to appropriate Sobolev norm has the structure of infinitely dimensional Banach orbifold groupoid.

We start from some general discussion. Let $C = (C_0, C_1)$ and $G = (G_0, G_1)$ be two orbifold groupoids. Let M_0 to be the set of groupoid morphisms from C to G . Let $f = (f_0, f_1) \in M_0$. We have the bundle $f^*TG \rightarrow C$. The neighborhood of f in M_0 can be identified with the neighborhood of 0-sections in $\Gamma(f^*TG)$, the space of sections of the bundle $f^*TG \rightarrow C$. After completed with respect to an appropriate Sobolev norms, M_0 is a Banach manifold.

M_0 has a natural equivalence relation by the natural transformations from C to G . It defines the set of arrows M_1 . Now, we show that an

arrow acts as a local diffeomorphism and hence (M_0, M_1) is an orbifold groupoid.

Suppose that $\alpha : C_0 \rightarrow G_1$ is a natural transformation from morphism \mathbf{f} to \mathbf{f}' . Namely, $\alpha(x)(f_0(x)) = f'_0(x)$ and commutes with the actions of G_1 . Therefore,

$$(\mathbf{f}')^* \mathbf{T}G = \alpha^* \mathbf{f}^* \mathbf{T}G.$$

Hence, α induces an isomorphism between the space of sections and a local diffeomorphism of M_0 under an appropriate Sobolev norm. Let $\mathbf{M} = (M_0, M_1)$ be the space of equivalence class of groupoid morphisms. We have showed that

Lemma 5.1. $\mathbf{M}(\mathbf{C}, \mathbf{G}) := \text{Mor}(\mathbf{C}, \mathbf{G})$ endowed with an appropriate Sobolev norm is a Banach orbifold groupoid.

Now, we allow the refinement of \mathbf{C} to consider the space $\mathbf{O}(\mathbf{C}, \mathbf{G}) := \text{Orb}(\mathbf{C}, \mathbf{G})$ of the equivalence classes of orbifold morphisms. Fix an equivalence $\epsilon : \mathbf{C} \rightarrow \mathbf{C}'$. $\mathbf{M}(\mathbf{C}', \mathbf{G})$ with an appropriate Sobolev norm is a Banach orbifold groupoid. We use it as a coordinate chart. Let $E(\mathbf{G})$ be the set of equivalence $\mathbf{C} \leftarrow \mathbf{C}'$. The set of objects of $\mathbf{O}(\mathbf{C}, \mathbf{G})$ is defined as

$$O_0(\mathbf{C}, \mathbf{G}) = \bigcup_{\mathbf{C} \leftarrow \mathbf{C}' \in E(\mathbf{G})} M_0(\mathbf{C}', \mathbf{G}).$$

The set of arrows $O_1(\mathbf{C}, \mathbf{G})$ consists of \mathcal{R} -equivalences. We checked that a natural transformation induces a local diffeomorphism. We leave to the readers to check that the additional equivalence induces a local diffeomorphism as well. Hence

Lemma 5.2. *The groupoid*

$$\mathbf{O}(\mathbf{C}, \mathbf{G}) = (O_0(\mathbf{C}, \mathbf{G}), O_1(\mathbf{C}, \mathbf{G}))$$

endowed with an appropriate Sobolev norm has a structure of Banach orbifold groupoid.

5.2.4. *Local Kuranishi structure at top stratum.* The moduli problem can be casted as a *continuous family* of Fredholm system. By a Fredholm system we mean that we have

- a Banach orbifold groupoid bundle \mathcal{F} over a Banach orbifold groupoid \mathcal{B} .
- a Fredholm section s of the bundle.

A continuous family of Fredholm system relative to a base B is a family of $\mathcal{F}_b \rightarrow \mathcal{B}_b$ for each $b \in B$. Furthermore, the total spaces $\mathcal{F} = \bigcup_b \mathcal{F}_b, \mathcal{B} = \bigcup_b \mathcal{B}_b$ have structures of topological orbifold groupoid and the projection map is a groupoid morphism.

A standard fact is that if s transverses to 0-section at each fiber, the zero set \mathcal{M} of s is a continuous family of smooth orbifolds and hence a topological orbifold.

In our case, the parameter of B is (1) the domain curves j and (2) germs in $\mathcal{O}_{h,\ell}$.

Let $\mathcal{B}_{j,[f]}^{1,p,\alpha}$ be the space of α -exponential decay of $[f]$ -type relative orbifold morphisms. This is a Banach groupoid, denoted by (B_0, B_1) . For each $u \in B_0$, we define a fiber to be the completion of $\Omega^{0,1}(u^*TG)$ with respect to the $L^{p,\alpha}$ norm. This forms a bundle $E_0 \rightarrow B_0$.

Let $\eta \in B_1$, and $s(\eta) = u$, $t(\eta) = u'$. Note that for each $x \in C_0$:

$$u_0^*TG_0|_x = TG_0|_{u_0(x)}, \quad (u'_0)^*TG_0|_x = TG_0|_{u'_0(x)}.$$

$\eta(x)$ induces an \mathbb{C} -isomorphism between these two spaces. Hence it induces an isomorphism $\sigma(\eta)$ between $\Omega^{0,1}(u^*TG)$ and $\Omega^{0,1}((u')^*TG)$. This defines the arrow section. We get a bundle $\mathcal{E}_{j,[f]} = (E_0, \sigma)$ over $\mathcal{B}_{j,[f]}^{1,p,\alpha}$. Then $\bar{\partial}$ is a section of the bundle and the moduli space is $\bar{\partial}^{-1}(0)$. By the explanation in §5.2.3 and the standard Fredholm theory for ∂ , we have

Lemma 5.3. $(\mathcal{B}_{j,[f]}^{1,p,\alpha}, \mathcal{E}_{j,[f]}, \bar{\partial})$ is a Fredholm system.

As the system varies at least continuously with respect to the parameter $(j, [f])$, we have

Corollary 5.4. $\pi : \mathcal{B}_{h,\ell}^{1,p,\alpha} \rightarrow B$ is a continuous family of orbifold Banach groupoids. Let $\delta : \mathcal{E} \rightarrow B$ be the union $\mathcal{E}_{j,[f]}$. Then $(\mathcal{B}_{h,\ell}^{1,p,\alpha}, \mathcal{E}, \bar{\partial})$ is a family of Fredholm system.

The index of the system can be computed via the index theorem. The idea is exactly same as the smooth case. Let $u \in \mathcal{M}_{j,[f]}$ be the stable map using cylinder model, and \bar{u} be the corresponding stable map in (G, Z) . Then the index of the system of the family at u is same as that of the index of $\bar{\partial}$ at \bar{u} (cf. Proposition 5.3 [LR]). Therefore the index is same as the index of relative moduli space (cf. (4.3)).

Once we have a continuous family of Fredholm system, a standard stabilization construction produces the local Kuranishi structure for $\mathcal{M} \subset \mathcal{B}$. We explain the construction for general Fredholm system.

Let $\mathcal{F} \rightarrow \mathcal{B}$ be a Banach orbifold bundle. In terms of groupoids, suppose that $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ and $\mathcal{F} = (\mathcal{F}_0, \sigma)$. Let S be a section of $\mathcal{F}_0 \rightarrow \mathcal{B}_0$ that induces a section of the orbifold bundle. For $x \in \mathcal{B}_0$ let L_x be the linear operator given in Appendix A.1.1.

Let $x \in \mathcal{M}$. If L_x is surjective, then by the standard argument of transversality, there exists a small neighborhood $U \subset \mathcal{M}$ of x that is homeomorphic to an orbifold.

Now suppose that L_x is not surjective. Then we stabilize the system at x for the system $(\mathcal{F}_0, \mathcal{B}_0, S)$. Using (C1) and (C2) in [CT] (cf. §5.1 and §5.2 in [CT]) to stabilization the system at a neighborhood of x : let O^x be the space that isomorphic to the cokernel of L_x and consider the following equation

$$\tilde{S}_x(y, v) = S(y) + v, \quad y \in \mathcal{B}_0, v \in V.$$

There exists a small neighborhood U_x of x such that \tilde{S}_x is regular at $U_x \times O^x$. Here, we use a general notation. For example, in the current case $S = \bar{\partial}$. The construction may be done to be \mathcal{B}_1 invariant (namely, equivariantly with respect to the orbifold structure), hence we construct the Kuranishi structure for the Fredholm system $(\mathcal{B}, \mathcal{F}, S)$.

Remark 5.5. *We may require that for $f \in O^x$, its support is away from marked points on the domain.*

We construct a virtual neighborhood $(V_x, \mathcal{O}_x, \sigma_x)$. This is a local Kuranishi structure at x . We may project V_x to \mathcal{B}_0 by $V_x \subset U_x \times O^x \rightarrow U_x$, let the image be V'_x . Then $V'_x \cong V_x$. Some authors use $(V'_x, \mathcal{O}_x, \sigma_x)$ as a local Kuranishi structure.

5.3. Local Kuranishi structure for lower strata. When the dual graph Γ of a relative orbifold stable morphism \mathbf{u} has edges, or equivalently \mathbf{C} has nodes, we consider the corresponding stratum \mathcal{M}_Γ as a lower strata. The method of Fredholm system constructs a (local) Kuranishi structure for \mathcal{M}_Γ . However, our goal is to construct the Kuranishi structure for the entire moduli space. Then, an additional gluing construction is needed for this purpose. Such a gluing construction is not new. In our setting, the analytic aspect is the same as the smooth cases [LR] and this is explained in Appendix A. In this section, we focus on the construction with respect to the orbifold structure.

5.3.1. Gluing theorem for the case of absolute node. For simplicity, we only assume that the domain contains one nodal point. As a warm-up, we first consider the case that the nodal point is an absolute node.

Let \mathbf{u} be a stable map in \mathcal{M}_Γ . Suppose that the domain is

$$\mathbf{C} = \mathbf{C}^+ \wedge \mathbf{C}^-.$$

The nodal point is denoted by $y = y^+ = y^-$. Suppose that the orbifold structure at y is \mathbb{Z}_r for some integer r and the monodromy of the map is (g) . We denote $\mathbf{u} = (\mathbf{u}^+, \mathbf{u}^-)$.

Since \mathbf{C} is a degenerated symplectic orbifold (cf. Example 3.1 and §4.1.1). We have a family of curve $\mathbf{C}_t, t \in \mathfrak{D}_{r,\epsilon}$, that degenerates to

C. On the other hand, by forgetting the orbifold structure we have a family of curve $C_t, t \in \mathfrak{D}_{1,\epsilon}$ that degenerates to $|\mathbf{C}|$ as well.

If \mathbf{u} is a regular point in \mathcal{M}_Γ i.e., $\text{Coker} L_x = 0$, then the gluing theory asserts that for small ϵ and any $t \in \mathfrak{D}_{1,\epsilon}$, \mathbf{u} can be glued to a stable map u_t of C_t .

We sketch the construction of u_t . The construction consists of two steps: splicing \mathbf{u}^\pm to an almost holomorphic map v on \mathbf{C}_t , then perturbing v to a holomorphic one. The second step is a standard implicit function theorem argument which we summary in the appendix. Here, we focus on the splicing and, in particular, how the gluing parameter interchanges between \mathfrak{D}_1 and \mathfrak{D}_r .

For parameter $t \in \mathfrak{D}_1$, we glue two disks $\mathbb{D}^\pm \subset C^\pm$ to a cylinder, we denote it by C_t . We want to splice \mathbf{u}^\pm to be a map from C_t to \mathbf{G} . However, \mathbf{u}^\pm is defined on $\mathbb{D}^\pm = \tilde{\mathbb{D}}/\mathbb{Z}_r$ as orbifold morphisms. We should do the splicing on $\tilde{\mathbb{D}}$: suppose that

$$u_0^\pm : \tilde{\mathbb{D}}^\pm \rightarrow G_0,$$

we splice them with the gluing parameter $\tilde{t} \in \mathfrak{D}_r$, where $\tilde{t}^r = t$; by the parameter \tilde{t} , $\tilde{\mathbb{D}}^\pm$ glue to a cylinder $\tilde{\mathbf{C}}_{\tilde{t}}$; then splice u_0^\pm as the smooth case, we have $\mathbf{v}_{\tilde{t}} : \tilde{\mathbf{C}}_{\tilde{t}} \rightarrow \mathbf{G}$. Note that $\mathbf{C}_{\tilde{t}} \cong C_t$, $\mathbf{v}_{\tilde{t}}$ reduces to a map $v_t : C_t \rightarrow G_0$. It can be shown that v_t is independent of the choice of \tilde{t} .

We are now able to formulate the gluing theorem for absolute cases. For each $\mathbf{u} \in \mathcal{M}_\Gamma$, there are two lines \mathbb{C}^\pm over it (by forgetting the orbifold structure and taking the tangent space of nodal point in each component). They define two line bundles \mathbb{L}^\pm over \mathcal{M}_Γ . Then the gluing bundle is

$$\mathbb{L} := \mathbb{L}^+ \otimes \mathbb{L}^- \rightarrow \mathcal{M}_\Gamma.$$

Let Γ_0 be the stratum obtained by contracting the unique edge. The gluing theorem is stated as

Theorem 5.6. *Let $\mathbb{L} \rightarrow \mathcal{M}_\Gamma$ be the gluing bundle. For any precompact $\mathcal{U} \subset \mathcal{M}_\Gamma$, there exists a small constant $\epsilon = \epsilon(\mathcal{U})$ and a gluing map*

$$\Phi : \mathbb{D}_\epsilon^* \mathbb{L}|_{\mathcal{U}} \longrightarrow \mathcal{M}_{\Gamma_0}$$

such that it is injective and local homeomorphic.

The proof will be given in appendix.

5.3.2. *Gluing bundle at the case of relative nodes.* For simplicity, we suppose that

- (1) the target space is $\mathbf{G} = \mathbf{G}^+ \wedge_{\mathbf{Z}} \mathbf{G}^-$;
- (2) the domain curve \mathbf{C} consists of two components \mathbf{C}^\pm and relative nodal points are y_1, \dots, y_k with multiplicities r_1, \dots, r_k ;

(3) the contact orders at y_i are $\ell_i = k_i/r_i$.

Let Γ be a dual graph of a stable map $u : \mathbb{C} \rightarrow \mathbb{G}$ that consists of two parts $u^\pm : \mathbb{C}^\pm \rightarrow \mathbb{G}^\pm$. Let Γ^\pm be the dual graph for u^\pm . Then $\mathcal{M}_\Gamma(\mathbb{G})$ is the fiber product of $\mathcal{M}_{\Gamma^\pm}(\mathbb{G}^\pm, \mathbb{Z})$ with respect to the relative evaluation maps.

For each y_i , we have three bundles over $\mathcal{M}_\Gamma(M)$. We denote them by H_i , \tilde{H}_i and \mathcal{H}_i respectively. We explain them in order.

\mathcal{H}_i : there is a trivial bundle $\mathbb{N}^+ \otimes \mathbb{N}^- \rightarrow \mathbb{Z}$, we pull it back to $\mathcal{M}_\Gamma(M)$ via evaluation maps;

\tilde{H}_i : this is a line bundle induced from the degeneration of domain orbifold curve at the nodal point corresponding to y_i (cf. namely, the fiber is \mathbb{C} that contains \mathfrak{D}_{r_i} which appears in §4.1.1 and Lemma 4.2);

H_i : this is a line bundle induced from the degeneration of the domain by forgetting-orbifold-structure curve at the nodal point corresponding to i (cf. namely, the fiber is \mathbb{C} that contains \mathfrak{D}_1).

Among them, there are natural maps:

$$\tau_i : \tilde{H}_i \rightarrow \mathcal{H}_i, \quad \gamma_i : \tilde{H}_i \rightarrow H_i.$$

The first one is induced from the stable map at the nodal points. (Fiberwisely, the maps can be thought as $\tau_i(z) = z^{k_i}$, $\gamma_i(z) = z^{r_i}$).

Now we consider all nodes simultaneously. Set

$$H = \bigoplus_{i=1}^k H_i, \quad \tilde{H} = \bigoplus_{i=1}^k \tilde{H}_i, \quad \mathcal{H} = \bigoplus_{i=1}^k \mathcal{H}_i.$$

and $\tau = \oplus \tau_i$, $\gamma = \oplus \gamma_i$. Note that \mathcal{H}_i is trivial, hence

$$\mathcal{H}_i = \mathcal{M}_\Gamma \times \mathbb{C}_i, \quad \mathcal{H} = \mathcal{M}_\Gamma \times \bigoplus \mathbb{C}_i.$$

Let $\bar{\tau}_i : \tilde{H}_i \rightarrow \mathbb{C}_i$ be the projection and $\bar{\tau} = \oplus_i \bar{\tau}_i$.

Let $\Delta_{\mathbb{C}} \subset \oplus_i \mathbb{C}_i$ be the diagonal and set

$$\Delta\mathcal{H} := \mathcal{M}_\Gamma \times \Delta_{\mathbb{C}} \subset \mathcal{H}.$$

Note that $\Delta_{\mathbb{C}} \cong \mathbb{C}$ is *nothing but the complex plane \mathbb{C} of the parameter \mathfrak{D} of the family \mathcal{D}* . (cf. Proposition 4.1). Define

$$\tilde{H}_\Delta = \tau^{-1} \Delta\mathcal{H} = \bar{\tau}^{-1} \Delta_{\mathbb{C}}, \quad H_\Delta = \gamma \tilde{H}_\Delta.$$

We find that the fiber of \tilde{H}_Δ is isomorphic to

$$T_{k_1, \dots, k_k} := \{(z_1, \dots, z_k) \mid z_1^{k_1} = \dots = z_k^{k_k}\}.$$

This is a one dimensional curve. Its normalization is \mathbb{C} realized by

$$\mathbb{C} \rightarrow \mathbb{C}_1 \times \cdots \times \mathbb{C}_k; \quad w \rightarrow (w^{\frac{\kappa}{k_1}}, \dots, w^{\frac{\kappa}{k_k}}),$$

where $\kappa = \prod k_i$. Let $\iota : \tilde{\mathbb{H}} \rightarrow \tilde{H}_\Delta$ be the fiberwise normalization of \tilde{H}_Δ . Define

$$(5.1) \quad \mathbb{H} = \frac{\tilde{\mathbb{H}}}{\mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_k}}.$$

It is called the *gluing bundle* of \mathcal{M}_Γ .

Remark 5.7. *The true gluing bundle is H_Δ . By the construction, \mathcal{H} is the normalization of H_Δ . In other word, $\mathbb{H}^* \cong (H_\Delta)^*$.*

Consider the map

$$(5.2) \quad \bar{\iota} : \tilde{\mathbb{H}} \xrightarrow{\iota} \tilde{H}_\Delta \xrightarrow{\bar{\lambda}} \Delta_{\mathbb{C}}.$$

When $\bar{\iota}$ restricts on the fiber of $\tilde{\mathbb{H}}$, the mapping degree is $\kappa = \prod k_i$. Hence the "mapping degree" from the fiber of \mathbb{H} to $\Delta_{\mathbb{C}}$ is $\ell = \prod \ell_i$.

5.3.3. *The gluing theorem at the case of the relative nodes.* We now state the gluing theorem. Let Γ_0 be the graph obtained by contracting all relative edges.

Theorem 5.8. *Suppose that \mathcal{M}_Γ is regular. For any precompact $\mathcal{U} \subset \mathcal{M}_\Gamma$, there exists a small constant $\epsilon = \epsilon(\mathcal{U}) > 0$ and a gluing map*

$$\Phi : \mathbb{D}_\epsilon^* \mathbb{H}|_{\mathcal{U}} \rightarrow \mathcal{M}_{\Gamma_0}$$

that is injective and local homeomorphic.

The detail of the proof of the analysis will be given in appendix. Here, we sketch the construction of the gluing map. Again, it consists of two crucial steps: construct the splicing map Ψ and right inverses. Since we work on the punctured disk bundle (cf. Remark 5.7)

$$\tilde{\mathbb{H}}^* \cong \tilde{H}_\Delta^*, \quad \mathbb{H}^* \cong H_\Delta^*.$$

Suppose that we have a point $\xi \in \mathbb{H}^*$ on the fiber over the stable map $\mathbf{u} = (\mathbf{u}^+, \mathbf{u}^-)$ defined on the domain curve \mathbb{C} . Let $\tilde{\xi} \in \tilde{\mathbb{H}}$ be the preimage of ξ with respect to γ . We explain how to get $\Psi(\xi)$. We should determine

- the domain curve;
- the target space;
- the almost holomorphic map.

Here, we explain these step by step.

The domain curve. Suppose the fiber coordinate of $\tilde{\xi}|_{\mathbf{u}}$ is $s = (s_1, \dots, s_k)$. We deform \mathbf{C} at i -th nodal point with parameter s_i . The resultant surface is $\mathbf{C}_{\tilde{\xi}}$; on the other hand, forgetting the orbifold structure at nodes, we deform C with respect to ξ , we denote the curve C_{ξ} . It is easy to see that $\mathbf{C}_{\tilde{\xi}} \cong C_{\xi}$. (cf. the end of §4.1.1).

The target space. The target space is \mathbf{G}_t with $t = \bar{t}(\tilde{\xi})$.

The almost holomorphic map. After the first two steps, the splicing is routine (cf. §4 [LR]). We use cut-off functions to splice \mathbf{u}^{\pm} to get an almost holomorphic map $\mathbf{v} : \mathbf{C}_{\tilde{\xi}} \rightarrow \mathbf{G}_t$. Set $\mathbf{v} = \Psi(\mathbf{u})$.

The right inverse: we use the regularity of \mathbf{u} to get the right inverse $Q_{\mathbf{u}}$ to $D_{\mathbf{u}}$, hence a right inverse to $D_{\mathbf{v}}$. The construction is explained in [LR] (see Lemma 4.8, [LR]). We denote the right inverse $Q_{\mathbf{v}}$.

The stable map: Then by the Taubes' argument (cf. Proposition A.3), we can perturb this map to a holomorphic map. This completes the construction of Φ .

5.3.4. The local Kuranishi structure of lower strata. Let $x \in \mathcal{M}_{\Gamma}$. We may construct the local Kuranishi structure within the stratum as we did for the top stratum. However, we should construct the structure for the entire moduli space. This can be done with the aid of the gluing theorem.

Let $(V_{x,\Gamma}, \mathcal{O}_{x,\Gamma}, \sigma_{x,\Gamma})$ be a local Kuranishi structure within the stratum. We still have the gluing bundle \mathbb{H} over $V_{x,\Gamma}$. $V_{x,\Gamma}$ plays the role as \mathcal{M}_{Γ} in Theorem 5.8. Suppose $(u, p) \in V_{x,\Gamma}$ satisfies the equation $\bar{\partial}u + p = 0$. The gluing bundle is still \mathbb{H} . This is a slightly more general situation than §5.3.3. However, we can construct the perturbation p such that $p = 0$ near marked and nodal points. Then, the equation is $\bar{\partial} = 0$ near the nodal points. The argument in §5.3.3 still applies. Hence, we construct a neighborhood V_x . $(V_x, V_x \times O^x, \sigma^x)$ gives a local Kuranishi structure.

5.4. Virtual fundamental cycles and relative invariants.

5.4.1. Patching. To construct a global Kuranishi structure, we still need to patch the local Kuranishi structures together. The patching argument is standard. The strata in $\overline{\mathcal{M}}$ has the partial order (cf. §2.2.2). In [FO], they started from the lowest strata and constructed Kuranishi structure inductively. We can mimic the argument in §15([FO]) and

hence obtain a global Kuranishi structure. The argument is a direct copy of that of [FO] and we omit it.

The general theory of Kuranishi structure implies the existence of a virtual fundamental cycle for $\overline{\mathcal{M}}$. Therefore, we conclude that

Theorem 5.9. *The global Kuranishi structure for $\overline{\mathcal{M}}_\Gamma(\mathbf{G}, \mathbf{Z})$ exists. Hence, we have a virtual fundamental cycle $[\overline{\mathcal{M}}_\Gamma(\mathbf{G}, \mathbf{Z})]^{vir}$.*

Similarly, we may construct the Kuranishi structure with boundary for $[\overline{\mathcal{M}}_\Gamma(\mathcal{D})]^{vir}$. The boundary is $[\lambda^{-1}(\partial D_o)]^{vir}$.

5.4.2. *Orbifold relative Gromov-Witten invariants.* Let

$$\overline{\mathcal{M}} := \overline{\mathcal{M}}_{g,(\mathbf{g}),A,(\mathbf{h}),T_k}(\mathbf{G}, \mathbf{Z})$$

be a relative moduli space. We have evaluation maps:

$$ev_i : \overline{\mathcal{M}} \rightarrow \mathbf{G}_{(g_i)}, \quad 1 \leq i \leq m; \quad ev_j^r : \overline{\mathcal{M}} \rightarrow \mathbf{G}_{(h_j)}, \quad 1 \leq j \leq k.$$

Then for

$$\alpha_i \in H^*(\mathbf{G}_{(g_i)}), \quad 1 \leq i \leq m, \quad \beta_j \in H^*(\mathbf{Z}_{(h_j)}), \quad 1 \leq j \leq k$$

we define a relative Gromov-Witten invariant by

$$(5.3) \quad \langle \prod_{i=1}^m \tau_{l_i} \alpha_i | \mathcal{T}_k \rangle_g = \frac{1}{|Aut(\mathcal{T}_k)|} \int_{[\overline{\mathcal{M}}]^{vir}} \prod_{i=1}^m ev_i^*(\alpha_i) \psi_i^{l_i} \prod_{j=1}^k (ev_j^r)^*(\beta_j).$$

Here

$$\mathcal{T}_k = ((\ell_1, h_1, \beta_1), \dots, (\ell_k, h_k, \beta_k)).$$

Recall that an orbifold structure is a orbifold Morita equivalence of orbifold groupoid. It is easy to check that all our constructions are preserved under orbifold Morita equivalence. Hence,

Theorem 5.10. *The virtual fundamental cycle $[\overline{\mathcal{M}}]^{vir}$ and the relative invariants $\langle \prod_{i=1}^m \tau_{l_i} \alpha_i | \mathcal{T}_k \rangle_g$ are independent of a particular orbifold groupoid presentation and invariants of the underline orbifold structure.*

6. THE DEGENERATION FORMULA

In this section, we give the degeneration formula for orbifold Gromov-Witten invariants. For $\mathbf{G} = \mathbf{G}^+ \wedge_{\mathbf{Z}} \mathbf{G}^-$, let \mathbf{G}_t be a generic fiber of the family $\mathcal{D} \rightarrow \mathfrak{D}$. The degeneration formula is in the form

$$GW(\mathbf{G}_t) = GW(\mathbf{G}^+, \mathbf{Z}) * GW(\mathbf{G}^-, \mathbf{Z}).$$

In this section, we use $\overline{\mathcal{M}}(\mathcal{D})$ to build a bridge connecting two sides.

Suppose that the moduli space is $d + 2$ -dimensional. Recall that we have the family

$$\lambda : \overline{\mathcal{M}}_{g,(\mathbf{g}),[A]}(\mathcal{D}) \rightarrow \mathfrak{D}.$$

For each $t \in \mathfrak{D}$,

$$[\lambda^{-1}(t)]^{vir} = [\overline{\mathcal{M}}_{g,(\mathbf{g}),[A]}(\mathbf{G}_t)]^{vir}$$

is a d -cycle.

To simplify the notation, we assume the regularity on $\overline{\mathcal{M}}_{g,(\mathbf{g}),[A]}(\mathcal{D})$ such that it is a topological orbifold of expected dimension. Otherwise, we work with the Kuranishi structure and the argument is essentially same. Consider the top strata of $\lambda^{-1}(0)$. Let $\mathcal{M}_\Gamma(\mathbf{G})$ be a component. Let \mathbf{u} be a point in the stratum. At the moment, we assume that the domain curve is

$$\mathbf{C} = \mathbf{C}^+ \wedge_y \mathbf{C}^-$$

and the stable map is $\mathbf{u} = (\mathbf{u}^+, \mathbf{u}^-) : \mathbf{C} \rightarrow \mathbf{G}$. Let Γ^\pm be the dual graph of \mathbf{u}^\pm . Then $\mathcal{M}_\Gamma(\mathbf{G})$ is the fiber product of $\mathcal{M}_{\Gamma^\pm}(\mathbf{G}^\pm, \mathbf{Z})$ with respect to the relative evaluation maps. Let ℓ be the contact order at y .

By the gluing theorem, we conclude that *the neighborhood of $\mathcal{M}_\Gamma(\mathbf{G})$ is isomorphic to the disk-bundle $\mathbb{D}_\epsilon \mathbb{H}_\Gamma$, where \mathbb{H}_Γ is the gluing bundle over $\mathcal{M}_\Gamma(\mathbf{G})$ (see the defining formula (5.1)).* Consider the map

$$\mathbb{D}_\epsilon \mathbb{H}_\Gamma \xrightarrow{\Phi} \mathcal{M}_{g,(\mathbf{g}),[A]}(\mathcal{D}) \xrightarrow{\lambda} \mathfrak{D}.$$

Fiberwisely, the degree of the map of disk is ℓ (cf. the end of §5.3.2). Hence, we conclude that

Lemma 6.1. *For any small $t \neq 0$,*

$$[\lambda^{-1}(t)]^{vir} = \sum_{\Gamma} \ell(\Gamma) [\mathcal{M}_\Gamma(\mathbf{G})]^{vir}.$$

where $\ell(\Gamma)$ is the product all contact orders at relative nodes.

On the other hand, it is routine to relate $[\mathcal{M}_\Gamma(\mathbf{G})]$ with $[\mathcal{M}_{\Gamma^\pm}(\mathbf{G}^\pm, \mathbf{Z})]$. In fact, (for simplicity, again we assume that there is only one relative node with monodromy (h))

$$[\mathcal{M}_\Gamma(\mathbf{G})]^{vir} = [\mathcal{M}_{\Gamma^+}(\mathbf{G}^+, \mathbf{Z}) \times_{\mathbf{Z}_{(h)}} \mathcal{M}_{\Gamma^-}(\mathbf{G}^-, \mathbf{Z})]^{vir}$$

Since the lower strata on the both side are of codimension at least 2, we have

$$(6.1) \quad [\overline{\mathcal{M}}_\Gamma(\mathbf{G})]^{vir} = [\overline{\mathcal{M}}_{\Gamma^+}(\mathbf{G}^+, \mathbf{Z}) \times_{\mathbf{Z}_{(h)}} \overline{\mathcal{M}}_{\Gamma^-}(\mathbf{G}^-, \mathbf{Z})]^{vir}$$

From Lemma 6.1 and (6.1), it is routine to formulate the degeneration formula.

Recall that we have

$$\phi_* : H_2(\mathbf{G}_t) \rightarrow H_2(\mathbf{G}), \quad \phi^* : H_{CR}^*(\mathbf{G}) \rightarrow H_{CR}^*(\mathbf{G}_t).$$

On the other hand, for $\alpha^\pm \in H_{CR}^*(\mathbf{G}^\pm)$ with $\alpha^+|_{\wedge Z} = \alpha^-|_{\wedge Z}$, it defines a class on $H_{CR}^*(\mathbf{G})$ which is denoted by (α^+, α^-) . Let $\alpha = \phi^*(\alpha^+, \alpha^-)$.

Theorem 6.2. *Suppose that \mathbf{G} is a degeneration of \mathbf{G}_t . Then*

$$(6.2) \quad \sum_{A \in [A]} \langle \mathbf{a} \rangle_{g,m,\mathbf{g},A}^{\bullet \mathbf{G}_t} = \sum_{\Gamma} \sum_I C(\Gamma, I) \langle \mathbf{a}^+ | \mathbf{b}^I \rangle_{\Gamma^+}^{\bullet(\mathbf{G}^+, Z)} \langle \mathbf{a}^- | \mathbf{b}_I^* \rangle_{\Gamma^-}^{\bullet(\mathbf{G}^-, Z)}.$$

where $C(\Gamma, I) = \ell(\Gamma) |\text{Aut}(\mathcal{T}(\Gamma, b^I))|$.

Notations in the formula are explained in order.

- (1) by \bullet , we mean the summation is on the non-connected dual graphs;
- (2) Γ runs over all the possible component in Lemma 6.1 and Γ^\pm is defined accordingly;
- (3) Let (b_1, \dots, b_k) be a basis of $H_{CR}^*(Z)$ and (b^1, \dots, b^k) be its dual basis. For $I \in \mathbb{Z}^n$, where n is number of relative nodes (or relative edges in Γ), say $I = (i_1, \dots, i_n)$ we define

$$b^I = b^{i_1} \wedge \dots \wedge b^{i_n}, \quad b_I = b_{i_1} \wedge \dots \wedge b_{i_n}.$$

- (4) given b^I , suppose the relative data for the relative nodes are $(\ell_j, (h_j), b^{i_j})$, then

$$\mathcal{T}(\Gamma, b^I) = \{(\ell_1, (h_1), b^{i_1}), \dots, (\ell_n, (h_n), b^{i_n})\}.$$

Proof: It is a direct consequence of Lemma 6.1 and formula (6.1).
q.e.d.

APPENDIX A. GLUING

The construction of Kuranishi structure at lower strata relies on the gluing construction (Theorem 5.6, Theorem 5.8). There is a huge amount of literatures on this topics. However, it is difficult to find a place where we can directly quote. In this appendix, we briefly outline the construction we need.

We start from the abstract set-up. Let $(\mathcal{B}, \mathcal{F}, \mathbf{s})$ be a Fredholm system for the moduli space \mathbf{M} . For simplicity, we *may* assume that $\mathcal{F} \rightarrow \mathcal{B}$ is a Banach bundle over Banach manifold. Otherwise, suppose that $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$, $\mathcal{F} = (\mathcal{F}_0, \sigma)$. Then we consider the system $(\mathcal{B}_0, \mathcal{F}_0, s_0)$. Let M_0 be the moduli space of this system. Once the coordinate chart is constructed on M_0 , it is easy to induce the coordinate chart on $\mathbf{M} = (M_0, M_1)$ as long as the construction is \mathcal{B}_1 invariant. Hence, the study is reduced to the study of the system $(\mathcal{B}_0, \mathcal{F}_0, s_0)$.

A.1. Set-up. The gluing theorem can be phrased as a construction of coordinate charts for M .

A.1.1. Pre-coordinate charts. For each $x \in \mathcal{B}$ if the linear operator given by

$$L_x : T_x \mathcal{B} \xrightarrow{ds_x} T_x \mathcal{F} \xrightarrow{\text{projection}} \mathcal{F}_x \cong F$$

is surjective, we call x a regular point. It is well known that if all points in M are regular, M is a smooth d -manifold. In this subsection, we explain how to construct coordinate charts for M .

Constructing the coordinate charts is a local problem. Hence, we assume the Fredholm system to be

$$(A.1) \quad (W, W \times F, s).$$

where W is a small neighborhood of 0 in a Banach space B . The section s defines a map $t : W \rightarrow F$ such that s is the graph of t . Then L_x is nothing but the tangent map of t at x .

Definition A.1. Suppose that the system (A.1) is given. Let M be the moduli space of the system and suppose that it is regular. For any $x \in M$, if we have

- (1) a smooth sub-manifold X in W ,
- (2) a small open ball $B_\delta \subset F$, a neighborhood U of x in W and a diffeomorphism

$$\Phi : X \times B_\delta \rightarrow U,$$

- (3) a smooth section

$$f : X \rightarrow B_\delta$$

such that the map given by

$$\mathbf{f} : X \xrightarrow{(1,f)} X \times B_\delta \xrightarrow{\Phi} U$$

maps X onto $U \cap M$ and the map is diffeomorphic, we then call (X, Φ, \mathbf{f}) (or (X, Φ, f) , if no confusion may be caused,) a pre-coordinate chart. In fact,

$$\mathbf{f}^{-1} : U \cap M \rightarrow X$$

gives a coordinate chart of M .

A.1.2. Assumptions. We make the following assumption on the system.

Assumption A.1 (Uniform continuity up to 2nd order). Suppose that W is bounded, i.e, there exists a constant $K_1 > 0$ such that $\|W\|_B \leq K_1$. Then there exists a constant C_1 that depends only on K_1 such that

(B1) for any $x, y \in W$

$$\|t(x) - t(y)\| \leq C_1 \|x - y\|_B;$$

(B2) for any $x, y \in W$

$$\|L_x - L_y\| \leq C_1 \|x - y\|_B;$$

(B3) for $x \in W$, N_x , defined by

$$N_x(\xi) = t(x + \xi) - t(x) - L_x \xi,$$

satisfies

$$\|N_x(\xi_1) - N_x(\xi_2)\| \leq C_1(\|\xi_1\|_B + \|\xi_2\|_B)(\|\xi_1 - \xi_2\|_B).$$

Again, we start with the system (A.1). Let X be a d -smooth submanifold of W . Assume that all points of X are regular. Over X , there is a bundle \mathcal{Q} of right inverses, i.e, the fiber over $x \in X$ is the space of right inverses of L_x . Let \mathbf{Q} be a smooth section of \mathcal{Q} . Set $Q_x = \mathbf{Q}(x)$. Then we can define a map

$$(A.2) \quad \Phi : X \times L \rightarrow B; \quad \Phi(x, \eta) = x + Q_x \eta.$$

Recall that B is the Banach space containing W . We put the following assumptions on (X, \mathbf{Q}) .

Assumption A.2. Let $\delta_1, \epsilon_1 > 0$ be small constants and $C_2 > 0$ be a constant such that $\epsilon_1 \ll \delta_1 \ll C_2$. The pair (X, \mathbf{Q}) satisfies

(C1) W is a bounded by K_1 ;

(C2) the image of $\Phi(X \times B_\delta)$ is contained in W ;

(C3) for any $x \in X$

$$\|t(x)\|_F \leq \epsilon_1;$$

(C4) for any $x \in X$ and $\zeta \in T_x X$

$$\|L_x \zeta\|_F \leq \epsilon_1 \|\zeta\|_B;$$

(C5) for any $x \in X$

$$\|Q_x\| \leq C_2;$$

(C6) for any two points $x_i, i = 1, 2$ in X ,

$$\|Q_{x_1} - Q_{x_2}\| \leq C_2 \|x_1 - x_2\|_B;$$

The condition (C3)-(C4) roughly says that X approximates the moduli space, while (C5)-(C6) asserts the natural continuity of \mathbf{Q} .

The following statement is due to Taubes.

Proposition A.3. Let (X, \mathbf{Q}) be pair such that Assumption A.2 is satisfied. There exists a smooth map

$$f : X \rightarrow B_\delta$$

such that $x + Q_x f(x) \in M$. Conversely, any point $y \in M \cap \Phi(X \times B_{\delta_1})$ in the form $x + Q_x \xi, \xi \in B_\delta$ is given by $\xi = f(x)$. Moreover

$$(A.3) \quad \|f(x)\|_F \leq 2\epsilon_1.$$

We remark that we may assume that $\epsilon_1 \ll \delta \ll C_2^{-1}$.

Hence we have construct a map from

$$\psi : X \rightarrow M.$$

A.1.3. Approximation pair of coordinate charts.

Definition A.2. A pair (X, \mathcal{Q}) that satisfies Assumption A.2 is called a local approximation pair of local coordinate chart. It is called an approximation pair if the map ψ is injective.

Theorem A.4. Under Assumption A.2, an approximation pair of coordinate chart yields a pre-coordinate chart.

Proof. Let (X, \mathcal{Q}) be an approximation pair. Let x_o be a point in X . We claim: there exists a small neighborhood $X_o \subset X$ of x_o , a small constant δ and a neighborhood $U \subset W$ of x_o such that Φ given in (A.2) gives a diffeomorphism

$$\Phi : X_o \times B_\delta \rightarrow U.$$

Here δ depends only on C_2 and K_1 .

Verification of the claim. We may identify B with $\ker L_{x_o} \oplus F$ via

$$\xi + Q_{x_o}\eta \leftrightarrow (\xi, \eta).$$

With this identification, we rewrite the map Φ as

$$\begin{aligned} \Phi : X \times F &\rightarrow B \cong \ker L_{x_o} \oplus F; \\ \Phi(x, \eta) &= (\bar{x} + Q_x\eta - Q_{x_o}L_{x_o}(\bar{x} + Q_x\eta), L_{x_o}(\bar{x} + Q_x\eta)), \end{aligned}$$

where $\bar{x} := x - x_o$. The tangent map of Φ at (x, η) is

$$D\Phi_{(x, \eta)}(\zeta, \xi) = \begin{pmatrix} \zeta + I_{11} & I_{12} \\ I_{21} & \xi + I_{22} \end{pmatrix},$$

where

$$\begin{aligned} I_{11} &= \frac{dQ_x}{d\zeta}\eta - Q_{x_o}L_{x_o}\zeta - Q_{x_o}L_{x_o}\frac{dQ_x}{d\zeta}\eta =: I_{111} + I_{112} + I_{113}; \\ I_{12} &= Q_x\xi - Q_{x_o}L_{x_o}Q_x\xi; \\ I_{21} &= L_{x_o}(\zeta + \frac{dQ_x}{d\zeta}\eta); \\ I_{22} &= L_{x_o}Q_x\xi - \xi. \end{aligned}$$

Note that $\zeta \in T_x X$. By direct estimates (using (C3)-(C5) in Assumption A.2), we have

$$|I_{111}| \leq C_2\delta, \quad |I_{112}| \leq C_2\epsilon_1, \quad |I_{113}| \leq C_2^2C_1\delta;$$

$$|I_{12}| \leq C_2^2 C_1 \|x - x_o\|, \quad |I_{21}| \leq \epsilon_1 + C_1 C_2 \delta, \quad |I_{22}| \leq C_1 C_2 \|x - x_o\|.$$

We can choose δ , X_o and ϵ such that

$$\|I_{ij}\| \leq \frac{1}{100} \|(\xi, \zeta)\|.$$

Hence by the dimension reason, $D\Phi_{(x,\eta)}$ is invertible and

$$(A.4) \quad \|D\Phi_{x,\eta}\| \leq 2, \quad (x, \eta) \in X \times B_\delta.$$

Now we show that Φ is injective. Suppose that

$$\Phi(x_1, \eta_1) = \Phi(x_2, \eta_2).$$

This says that in B

$$(A.5) \quad x_1 + Q_{x_1}\eta_1 = x_2 + Q_{x_2}\eta_2,$$

The expansion of $x + Q_x\eta$ at $(x_o, 0)$ is

$$x + Q_x\eta = x_o + [\bar{x} + Q_{x_o}\eta] + \mathcal{N}(x, \eta),$$

where as a higher order term

$$\mathcal{N}(x, \eta) = Q_x\eta - Q_{x_o}\eta.$$

The term in the square bracket is the linear term, we denote it by $P(x - x_o, \eta)$. Equation (A.5) then implies that

$$P(x_1 - x_2, \eta_1 - \eta_2) = \mathcal{N}(x_2, \eta_2) - \mathcal{N}(x_1, \eta_1).$$

We find that

$$\|\mathcal{N}(x_2, \eta_2) - \mathcal{N}(x_1, \eta_1)\| \leq C_2(\|\bar{x}_1, \eta_1\| + \|\bar{x}_2, \eta_2\|)\|(x_2, \eta_2) - (x_1, \eta_1)\|$$

On the other hand, P (equivalent to $D\Phi_{x_o,0}$) is invertible. Hence

$$\|P(x_1 - x_2, \eta_1 - \eta_2)\| \geq (100C_2)^{-1}\|(x_2, \eta_2) - (x_1, \eta_1)\|.$$

When X_o and δ small, we get

$$(100C_2)^{-1}\|(x_2, \eta_2) - (x_1, \eta_1)\| \leq (10000C_2)^{-1}\|(x_2, \eta_2) - (x_1, \eta_1)\|.$$

This is possible only when $(x_2, \eta_2) = (x_1, \eta_1)$. We verify the injectivity. q.e.d.

A.2. Local homeomorphism of gluing maps. We prove the gluing theorem at here. There is no difference to get the main estimates from the smooth case. We quote the estimates from [LR].

Let $v = \Psi(\xi)$ where ξ is a point on the fiber of \mathbb{H} over u . Let Q_v be the right inverse to D_v (the construction is described in [LR]).

Lemma A.5 (Lemma 4.6[LR]). *Let $r = |\xi|$. There exists a constant C , depending only on \mathcal{U} , such that*

$$\|\bar{\partial}v\| \leq Cr^{1-\alpha}.$$

Moreover, for any path u_s in \mathcal{U} ,

$$\left\| \frac{d\bar{\partial}v_s}{ds} \right\| \leq Cr^{1-\alpha} \left\| \frac{du_s}{ds} \right\|.$$

Here $\|\cdot\|$'s are proper norms accordingly.

Lemma A.6 (Lemma 4.8[LR]). *Let $r = |\xi|$. There exists a constant C , depending only on \mathcal{U} , such that*

$$\|Q_v\| \leq C.$$

Moreover, for any path u_s in \mathcal{U} ,

$$\left\| \frac{dQ_{v_s}}{ds} \right\| \leq C \left\| \frac{du_s}{ds} \right\|.$$

Here $\|\cdot\|$'s are proper norms accordingly.

Proof of Theorem 5.8. Set P to be the image of Ψ and Q be the collection of $Q_v, v \in P$. Then Lemma A.5 and A.6 implies that: when r is small, (P, Q) is a local approximation pair of local coordinate charts. Since the gluing map Φ is constructed from (P, Q) that is same as the map ψ given at the end of §A.1.2. Hence, we prove the local homeomorphism of Φ .

The injectivity of Φ can be shown by controlling the energy of the maps and the local homeomorphism of Φ . This is standard. We skip it. q.e.d.

The proof to Theorem 5.6 is identical.

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